Lecture 7

PSEUDOINVERSES II

Computational methods and applications

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$\boxed{\text{OUTLINE}}$

- Review of definitions

- Properties of the Moore-Penrose pseudoinverse

- Pseudoinverses and linear equations

- Iterative computation of the pseudoinverse

- Effects of noise

- Regularization

Decomposition of object and image space

An arbitrary object can be written as

$$\mathbf{f} = \sum_{n=1}^{N} \alpha_n \mathbf{u}_n = \sum_{n=1}^{R} \alpha_n \mathbf{u}_n + \sum_{n=R+1}^{N} \alpha_n \mathbf{u}_n$$

$$\mathbf{f} = \mathbf{f}_{meas} + \mathbf{f}_{null}\,, \qquad \boldsymbol{\mathcal{H}}\mathbf{f}_{null} = 0\,, \qquad \mathbf{f}_{meas} \perp \mathbf{f}_{null}\,.$$

An arbitrary vector in image space can be represented as:

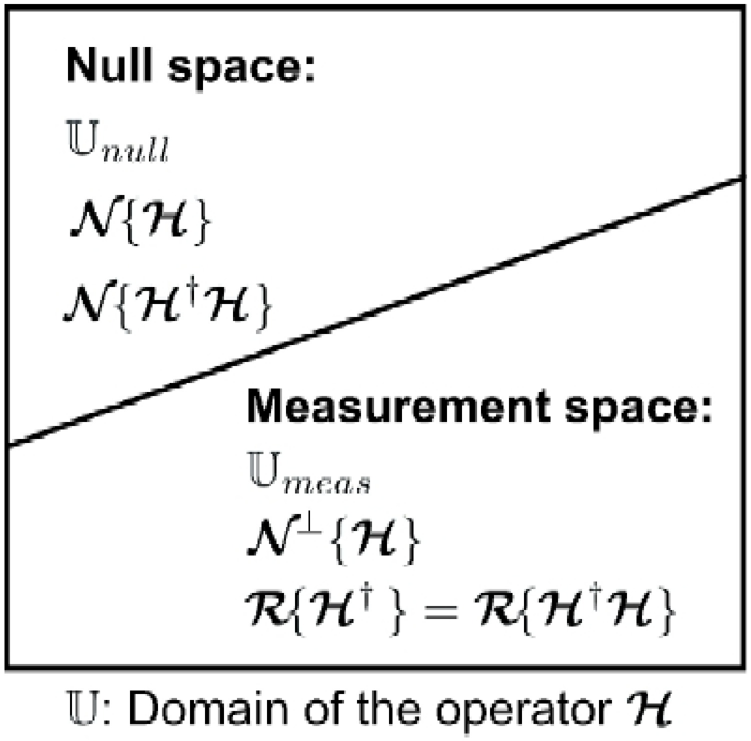
$$\mathbf{g} = \sum_{m=1}^{M} \beta_m \mathbf{v}_m = \sum_{m=1}^{R} \beta_m \mathbf{v}_m + \sum_{m=R+1}^{M} \beta_m \mathbf{v}_m$$

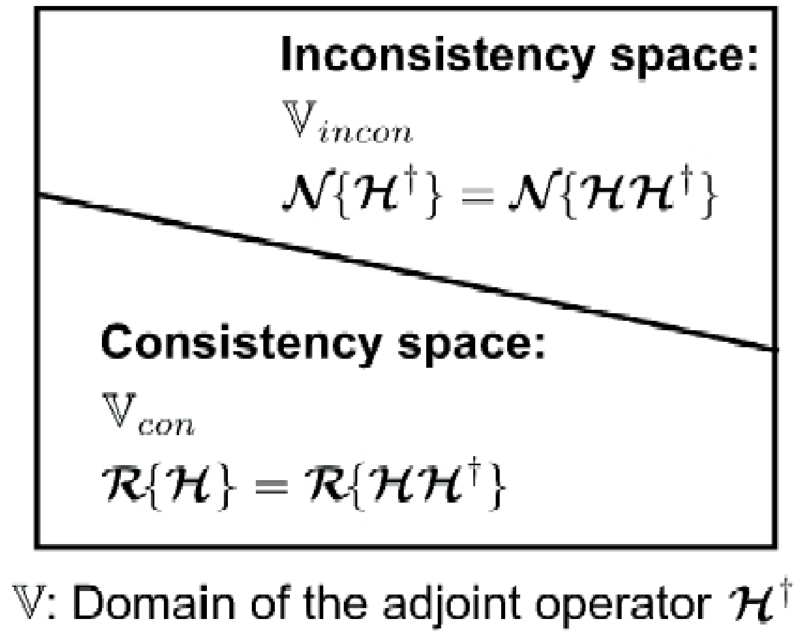
$$\mathbf{g} = \mathbf{g}_{con} + \mathbf{g}_{incon}\,, \qquad \boldsymbol{\mathcal{H}}^{\dagger}\mathbf{g}_{incon} = 0\,, \qquad \mathbf{g}_{con} \perp \mathbf{g}_{incon}$$

Thus we can decompose object and image space into orthogonal subspaces:

$$\mathbb{U} = \mathbb{U}_{meas} \oplus \mathbb{U}_{null}\,, \qquad \mathbb{V} = \mathbb{V}_{con} \oplus \mathbb{V}_{incon}$$

Four subspaces

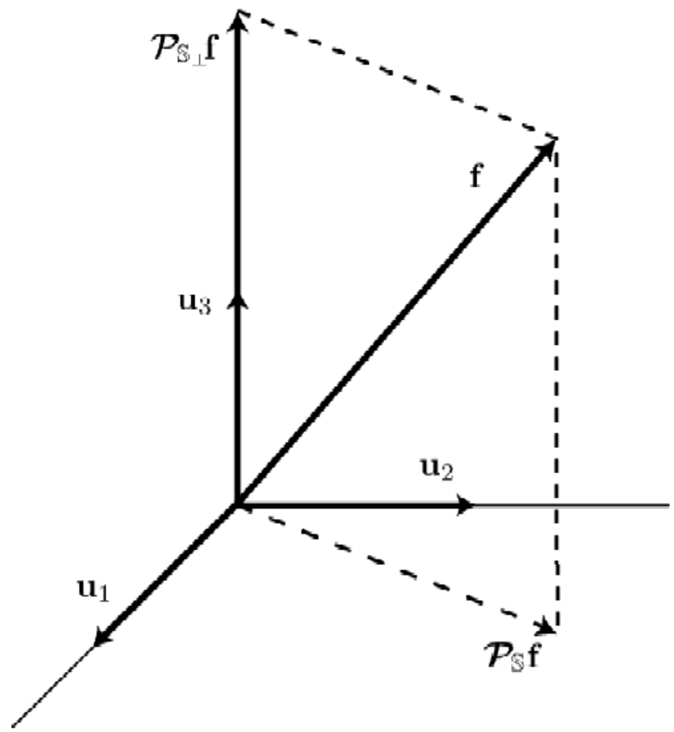




Recall $R \leq min(M, N)$. If $R = N$, there is no nontrivial null space. If $R = M$, there is no nontrivial inconsistency space.

Projection operators

A general projection operator $\mathcal{P}$ is Hermitian ($\mathcal{P}^\dagger = \mathcal{P}$) and *idempotent* ($\mathcal{P}^2 = \mathcal{P}$). We are interested here in projections onto subspaces:



Projections onto subspaces

$$\mathbf{f}_{meas} = \sum_{n=1}^{R} \alpha_n \mathbf{u}_n = \boldsymbol{\mathcal{P}}_{meas}\, \mathbf{f}\ , \qquad \mathbf{f}_{null} = \sum_{n=R+1}^{N} \alpha_n \mathbf{u}_n = \boldsymbol{\mathcal{P}}_{null}\, \mathbf{f}\ ,$$

where $\boldsymbol{\mathcal{P}}_{meas}$ and $\boldsymbol{\mathcal{P}}_{null}$ are the projectors onto measurement and null space, respectively. Can show that

$$\boldsymbol{\mathcal{P}}_{meas} = \sum_{n=1}^{R} \mathbf{u}_n \mathbf{u}_n^{\dagger}\,, \qquad \boldsymbol{\mathcal{P}}_{null} = \sum_{n=R+1}^{N} \mathbf{u}_n \mathbf{u}_n^{\dagger} = \mathbf{I}_{\mathbb{U}} - \boldsymbol{\mathcal{P}}_{meas}$$

Similarly,

$$\mathbf{g}_{con} = \sum_{m=1}^{R} \beta_m \mathbf{v}_m = \boldsymbol{\mathcal{P}}_{con}\, \mathbf{g}\ , \qquad \mathbf{g}_{incon} = \sum_{m=R+1}^{M} \beta_m \mathbf{v}_m = \boldsymbol{\mathcal{P}}_{incon}\, \mathbf{g}\ .$$

$$\boldsymbol{\mathcal{P}}_{con} = \sum_{m=1}^{R} \mathbf{v}_m \mathbf{v}_m^{\dagger}\,, \qquad \boldsymbol{\mathcal{P}}_{incon} = \sum_{m=R+1}^{M} \mathbf{v}_m \mathbf{v}_m^{\dagger} = \mathbf{I}_{\mathbb{V}} - \boldsymbol{\mathcal{P}}_{con}$$

Moore's definition of pseudoinverse

Moore (1920) defined an operator $\boldsymbol{\mathcal{H}}^+$ (a matrix in his case) such that:

$$\boldsymbol{\mathcal{P}}_{meas} = \boldsymbol{\mathcal{H}}^+\boldsymbol{\mathcal{H}}\,, \qquad \boldsymbol{\mathcal{P}}_{null} = \mathbf{I}_{\mathbb{U}} - \boldsymbol{\mathcal{H}}^+\boldsymbol{\mathcal{H}},$$

$$\boldsymbol{\mathcal{P}}_{con} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^+\,, \qquad \boldsymbol{\mathcal{P}}_{incon} = \mathbf{I}_{\mathbb{V}} - \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^+,$$

These four definitions turn out to completely determine $\boldsymbol{\mathcal{H}}^+$.

The Penrose equations and the Moore-Penrose pseusdoinverse

A *generalized inverse* is defined as any operator $\boldsymbol{\mathcal{H}}^{\#}$ that satisfies one or more of the following equations:

$$\text{Penrose Eq. 1: } \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\#}\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}\,. \tag{1.130a}$$

$$\text{Penrose Eq. 2: } \boldsymbol{\mathcal{H}}^{\#}\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\#} = \boldsymbol{\mathcal{H}}^{\#}\,. \tag{1.130b}$$

$$\text{Penrose Eq. 3: } (\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\#})^{\dagger} = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\#}\,. \tag{1.130c}$$

$$\text{Penrose Eq. 4: } (\boldsymbol{\mathcal{H}}^{\#}\boldsymbol{\mathcal{H}})^{\dagger} = \boldsymbol{\mathcal{H}}^{\#}\boldsymbol{\mathcal{H}}\,. \tag{1.130d}$$

The Moore-Penrose pseudoinverse, denoted $\boldsymbol{\mathcal{H}}^{+}$, is the generalized inverse that satisfies all four Penrose equations. It is equal to the true inverse if one exists.

The Moore-Penrose pseudoinverse exists for all operators with finite-dimensional range ($M < \infty$), including CD operators and matrices. It may not exist for some CC operators.

SVD expressions

Recall:

$$\boldsymbol{\mathcal{H}} = \sum_{n=1}^{R} \sqrt{\mu_n}\, \mathbf{v}_n \mathbf{u}_n^\dagger\,, \qquad \boldsymbol{\mathcal{H}}^\dagger = \sum_{n=1}^{R} \sqrt{\mu_n}\, \mathbf{u}_n \mathbf{v}_n^\dagger$$

Can define a pseudoinverse as

$$\boldsymbol{\mathcal{H}}^+ = \sum_{n=1}^{R} \frac{1}{\sqrt{\mu_n}}\, \mathbf{u}_n \mathbf{v}_n^\dagger$$

This operator satisfies both Moore and Penrose definitions.

Limiting representations

The Moore-Penroose pseudoinverse can be written in terms of true inverses in two different ways:

$$\boldsymbol{\mathcal{H}}^{+} = \lim_{\eta \to 0^{+}} \left[\boldsymbol{\mathcal{H}}^{\dagger}\boldsymbol{\mathcal{H}} + \eta\,\boldsymbol{\mathcal{I}}_{\mathbb{U}}\right]^{-1} \boldsymbol{\mathcal{H}}^{\dagger}\,. \tag{1.135}$$

$$\boldsymbol{\mathcal{H}}^{+} = \lim_{\eta \to 0^{+}} \boldsymbol{\mathcal{H}}^{\dagger} \left[\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^{\dagger} + \eta\,\boldsymbol{\mathcal{I}}_{\mathbb{V}}\right]^{-1}\,. \tag{1.141}$$

Term $\eta\boldsymbol{\mathcal{I}}$ guarantees that the indicated inverses exist.

See B&M for proof that the limits reproduce the SVD expressions.

These representations are useful for constructing practical iterative algorithms.

$\boxed{\text{Pseudoinverse indentities}}$

$$\left[\mathcal{H}^+\right]^+ = \mathcal{H}\,; \tag{1.147}$$

$$\mathcal{H}^+ = \left(\mathcal{H}^\dagger\mathcal{H}\right)^+\mathcal{H}^\dagger\,; \tag{1.148}$$

$$\mathcal{H}^+ = \mathcal{H}^\dagger\left[\mathcal{H}\mathcal{H}^\dagger\right]^+\,; \tag{1.149}$$

$$\left[\mathcal{H}^\dagger\right]^+ = \left[\mathcal{H}^+\right]^\dagger\,; \tag{1.150}$$

$$\left[\mathcal{H}^\dagger\right]^+ = \left[\mathcal{H}\mathcal{H}^\dagger\right]^+\mathcal{H}\,; \tag{1.151}$$

$$\left[\mathcal{H}^+\mathcal{H}\right]^+ = \mathcal{H}^+\mathcal{H}\,; \tag{1.152}$$

$$\mathcal{H}^+\mathcal{H}\mathcal{H}^\dagger = \mathcal{H}^\dagger\,; \tag{1.153}$$

$$\mathcal{H}^{\dagger}\mathcal{H}\mathcal{H}^{+} = \mathcal{H}^{\dagger}\,; \tag{1.154}$$

$$\mathcal{H}^{+}\mathcal{H} = \left(\mathcal{H}^{\dagger}\mathcal{H}\right)^{+}\left(\mathcal{H}^{\dagger}\mathcal{H}\right) = \left(\mathcal{H}^{\dagger}\mathcal{H}\right)\left(\mathcal{H}^{\dagger}\mathcal{H}\right)^{+}\;; \tag{1.155}$$

$$\mathcal{H}\mathcal{H}^{+} = \left(\mathcal{H}\mathcal{H}^{\dagger}\right)^{+}\mathcal{H}\mathcal{H}^{\dagger} = \mathcal{H}\mathcal{H}^{\dagger}\left(\mathcal{H}\mathcal{H}^{\dagger}\right)^{+}\;; \tag{1.156}$$

$$\left[\mathcal{H}^{\dagger}\mathcal{H}\right]^{+} = \mathcal{H}^{+}\left[\mathcal{H}^{\dagger}\right]^{+}\;; \tag{1.157}$$

$$\left[\mathcal{H}^{\dagger}\mathcal{H}\right]^{+} = \mathcal{H}^{+}\left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+}\mathcal{H} = \mathcal{H}^{\dagger}\left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+}\left[\mathcal{H}^{\dagger}\right]^{+}\;; \tag{1.158}$$

$$\left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+} = \left[\mathcal{H}^{\dagger}\right]^{+}\mathcal{H}^{+}\,. \tag{1.159}$$

All of these identities can be proved either directly from the Penrose equations or from the SVD representations of $\mathcal{H}$ and $\mathcal{H}^{+}$.

$\boxed{\text{Pseudoinverse of a product}}$

It is not true in general that $(\mathbf{AB})^{+} = \mathbf{B}^{+}\mathbf{A}^{+}$. Instead:

$$(\mathbf{AB})^{+} = (\mathbf{A}^{+}\mathbf{AB})^{+}(\mathbf{ABB}^{+})^{+}$$

Solutions of linear equations

Consider a matrix representation of an imaging system with noise:

$$g_m = \sum_{n=1}^{N} H_{mn} f_n + \epsilon_m \,, \qquad m = 1, ..., M \,, \tag{1.179}$$

where ϵ_m represents noise and modeling error. In matrix-vector form:

$$\mathbf{g} = \mathbf{H}\mathbf{f} + \boldsymbol{\epsilon} \,. \tag{1.180}$$

What is meant by a "solution" of this equation?

Mathematicians would define an "exact" solution as one that satisfies

$$\mathbf{g} = \mathbf{H}\hat{\mathbf{f}} \,. \tag{1.181}$$

- Does an exact solution exist?
- Is it unique?
- Is it stable?

$\boxed{\text{Existence}}$

$$\mathbf{g} = \mathbf{H}\hat{\mathbf{f}}\ . \tag{1.181}$$

Equation (1.181) has *at least* one exact solution for *a particular* $\mathbf{g}$ if and only if:

- The data are consistent;
- $\mathbf{g}$ lies in $\mathbb{V}_{con}$;
- $\mathbf{H}\mathbf{H}^{+}\mathbf{g} = \mathbf{g}$.

Equation (1.181) has *at least* one exact solution for *every* $\mathbf{g}$ if and only if:

- Rank $=$ number of measurements $(R = M)$;
- $\mathbb{V}_{incon}$ is trivial;
- $\mathbf{H}\mathbf{H}^{\dagger}$ is nonsingular.

These conditions can be satisfied only if $M \leq N$, i.e., the problem is *underdetermined*, more unknowns than measurements.

Uniqueness

$$\mathbf{g} = \mathbf{H}\hat{\mathbf{f}} \ . \tag{1.181}$$

Equation (1.181) has *at most* one solution for every **g** if and only if:

- Rank $=$ number of unknowns $(R = N)$;
- $\mathbb{U}_{null}$ is trivial;
- $\mathbf{H}^{\dagger}\mathbf{H}$ is nonsingular.

These conditions can be satisfied only if $M \geq N$ (i.e., the problem is overdetermined).

Putting the existence and uniqueness conditions together, we see that a unique, exact solution exists only if **H** is square and full rank, $M = N = R$.

An explicit solution for consistent data

A particular data vector is consistent if:

$$\mathbf{H}\mathbf{H}^{+}\mathbf{g} = \mathbf{g}\,, \tag{1.185}$$

which can happen in two ways:

- There is no nontrivial inconsistency space
- We cheated and generated noise-free data

If the data are consistent, one solution to (1.181) is

$$\hat{\mathbf{f}} = \mathbf{H}^{+}\mathbf{g}\,. \tag{1.186}$$

To show that this is a solution, operate on it with **H**. The result is

$$\mathbf{H}\hat{\mathbf{f}} = \mathbf{H}\mathbf{H}^{+}\mathbf{g} = \mathbf{g}\,, \tag{1.187}$$

where the last step follows from the consistency condition (1.185).

General solution for consistent data

$\mathbf{H}\hat{\mathbf{f}}$ is one solution to $\mathbf{g} = \mathbf{H}\hat{\mathbf{f}}$, but it is not unique if $\mathbf{H}$ has null vectors.

The most general solution is

$$\hat{\mathbf{f}} = \mathbf{H}^{+}\mathbf{g} + \left[\mathbf{I}_N - \mathbf{H}^{+}\mathbf{H}\right]\mathbf{y}\,, \tag{1.188}$$

where $\mathbf{y}$ is an arbitrary vector in $\mathbb{U}$. Since $\mathbf{I}_N - \mathbf{H}^{+}\mathbf{H}$ is just $\mathbf{P}_{null}$, the second term in (1.188) is a null vector of **H**.

To demonstrate that (1.188) is a solution to (1.181) for consistent data, we again operate on it with **H**, yielding

$$\mathbf{H}\hat{\mathbf{f}} = \mathbf{H}\mathbf{H}^{+}\mathbf{g} + \left[\mathbf{H} - \mathbf{H}\mathbf{H}^{+}\mathbf{H}\right]\mathbf{y}\,. \tag{1.189}$$

The first term is $\mathbf{g}$ by the consistency condition (1.185), while the second term is zero by the first Penrose equation, so (1.188) is indeed a solution to (1.181) if the data are consistent.

Least-squares solutions

For inconsistent data, no "exact" solution exists – there is no $\hat{\mathbf{f}}$ such that $\mathbf{H}\hat{\mathbf{f}} \equiv \mathbf{g}$.

One approach in that case is to look for a *least-squares solution* such that the norm of $\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}$ is minimized:

$$\hat{\mathbf{f}}_{LS} = \underset{\hat{\mathbf{f}}}{\text{argmin}} \; ||\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}||^2 \,. \tag{1.191}$$

In words, look for a solution that matches the data as closely as possible.

A statistical justification for this approach will be given next week.

Terminology: $\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}$ is called the *residual*

The normal equation

If there are no constraints (such as positivity) on the solution, a least-squares solution must satisfy:

$$\frac{\partial}{\partial \hat{f}_n} ||\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}||^2 = 0 \qquad \forall n \,.$$

Carrying out the derivative yields

$$\mathbf{H}^\dagger \mathbf{H} \hat{\mathbf{f}} = \mathbf{H}^\dagger \mathbf{g} \,. \tag{1.194}$$

This important result is called the *normal equation*. Solving the normal equation is equivalent to finding a least-squares solution to the original equation, $\mathbf{g} = \mathbf{H}\hat{\mathbf{f}}$.

Existence and uniqueness revisited

A solution to the normal equation always exists, but it is not unique if $\mathbf{H}$ has null vectors.

if $\mathbf{H}^\dagger\mathbf{H}$ is nonsingular or $R = N$, the least-squares solution is given uniquely by

$$\hat{\mathbf{f}}_{LS} = \left(\mathbf{H}^\dagger\mathbf{H}\right)^{-1}\mathbf{H}^\dagger\mathbf{g}\,, \qquad (R = N)\,. \tag{1.195}$$

With a null space ($R < N$), the general least-squares solution has exactly the same form as (1.188):

$$\hat{\mathbf{f}}_{LS} = \mathbf{H}^+\mathbf{g} + \left[\mathbf{I}_N - \mathbf{H}^+\mathbf{H}\right]\mathbf{y}\,, \tag{1.196}$$

where again **y** is an arbitrary vector in $\mathbb{U}$.

Minimum-norm solutions

General solution repeated:

$$\hat{\mathbf{f}}_{LS} = \mathbf{H}^{+}\mathbf{g} + \Big[\mathbf{I}_N - \mathbf{H}^{+}\mathbf{H}\Big]\mathbf{y}\,, \tag{1.196}$$

One way to get a unique LS solution is simply to demand that there be no null component and set $\mathbf{y} = 0$.

Note that this choice minimizes the norm *of the solution* (as well as the norm of the residual):

$$||\hat{\mathbf{f}}||^2 = ||\hat{\mathbf{f}}_{meas}||^2 + ||\hat{\mathbf{f}}_{null}||^2\,.$$

Thus the *minimum-norm, least-squares* (MNLS) solution is

$$\hat{\mathbf{f}}_{MNLS} = \mathbf{H}^{+}\mathbf{g}\,.$$

$\boxed{\text{Another view of MNLS}}$

A least-squares solution is given by

$$\hat{\mathbf{f}}_{LS} = \underset{\hat{\mathbf{f}}}{\text{argmin}} \; ||\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\,||^2 \,. \tag{1.191}$$

To get the unique MNLS solution, we can require

$$\hat{\mathbf{f}}_{MNLS} = \lim_{\eta \to 0} \; \underset{\hat{\mathbf{f}}}{\text{argmin}} \; \left\{ ||\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\,||^2 + \eta||\hat{\mathbf{f}}\,||^2 \right\} \,.$$

Since $\eta \to 0$, the extra term does not disrupt the LS requirement, but for any $\eta > 0$ it penalizes null functions.

Computation of the MP pseudoinverse

We now know that the MNLS solution to $\mathbf{g} = \mathbf{H}\hat{\mathbf{f}}$ always exists and is always unique. It is given formally by $\hat{\mathbf{f}}_{MNLS} = \mathbf{H}^{+}\mathbf{g}$, but how do we compute it?

Two possible answers:

- Do an SVD and use $\boldsymbol{\mathcal{H}}^{+} = \sum_{n=1}^{R} \frac{1}{\sqrt{\mu_n}} \mathbf{u}_n \mathbf{v}_n^{\dagger}$

- Find an iterative algorithm that converges to $\mathbf{H}^{+}\mathbf{g}$

The Landweber algorithm

Consider the following iteration:

$$\hat{\mathbf{f}}^{(k+1)} = \hat{\mathbf{f}}^{(k)} + \mathbf{H}^{\dagger}\left[\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}^{(k)}\right] . \tag{1.231}$$

If this algorithm converges, we must have

$$\hat{\mathbf{f}}^{(k+1)} = \hat{\mathbf{f}}^{(k)}$$

and hence

$$\mathbf{H}^{\dagger}\mathbf{g} = \mathbf{H}^{\dagger}\mathbf{H}\hat{\mathbf{f}}^{(k)} ,$$

which is just the normal equation. Moreover, the solution cannot contain any null functions, provided the initial estimate containt none, since the operator $\mathbf{H}^{\dagger}$ "erases" the null functions in the correction term. A suitable initial estimate is $\hat{\mathbf{f}}^{(0)} = \mathbf{H}^{\dagger}\mathbf{g}$. With this choice,

$$\hat{\mathbf{f}}^{(\infty)} = \hat{\mathbf{f}}_{MNLS}$$

For derivation of the Landweber algorithm and discussion of convergence, see B&M Sec. 1.7.6.

$\boxed{\text{Noise amplification}}$

If we represent the actual vector $\mathbf{f}$ (not its estimate) by

$$\mathbf{f} = \sum_{k=1}^{N} \alpha_k \mathbf{u}_k\,, \qquad \alpha_k = \mathbf{u}_k^\dagger \mathbf{f}\,, \tag{1.207}$$

and the noise by

$$\boldsymbol{\epsilon} = \sum_{k=1}^{M} \gamma_k \mathbf{v}_k\,, \qquad \gamma_k = \mathbf{v}_k^\dagger \boldsymbol{\epsilon}\,, \tag{1.208}$$

then the expansion coefficients for $\mathbf{g}$ are given by

$$\beta_k = \sqrt{\mu_k}\,\alpha_k + \gamma_k\,. \tag{1.209}$$

With the SVD form of $\mathbf{H}^+$, we find

$$\hat{\mathbf{f}}_{MNLS} = \sum_{k=1}^{R} \left[\alpha_k + \frac{\gamma_k}{\sqrt{\mu_k}}\right] \mathbf{u}_k = \mathbf{f}_{meas} + \sum_{k=1}^{R} \frac{\gamma_k}{\sqrt{\mu_k}}\,\mathbf{u}_k\,, \tag{1.210}$$

Noise in the pseudoinverse solution is very large if μ_k is near zero.

Tikhonov regularization

Recall

$$\hat{\mathbf{f}}_{MNLS} = \lim_{\eta \to 0} \underset{\hat{\mathbf{f}}}{\text{argmin}} \left\{ ||\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\,||^2 + \eta ||\hat{\mathbf{f}}\,||^2 \right\} .$$

One way to control the noise is just to omit the limit:

$$\hat{\mathbf{f}}_{RLS} = \underset{\hat{\mathbf{f}}}{\text{argmin}} \left\{ ||\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}\,||^2 + \eta ||\hat{\mathbf{f}}\,||^2 \right\} ,$$

where RLS stands for *regularized least squares*.

The RLS solution can be found by a modified Landweber algorithm:

$$\hat{\mathbf{f}}^{(k+1)} = (1 - \eta)\,\hat{\mathbf{f}}^{(k)} + \mathbf{H}^{\dagger} \left[\mathbf{g} - \mathbf{H}\hat{\mathbf{f}}^{(k)} \right] .$$

The Tikhonov RLS solution in SVD form is

$$\hat{\mathbf{f}}_{RLS} = \sum_{k=1}^{R} \frac{\mu_k \, \alpha_k}{\mu_k + \eta} \mathbf{u}_k + \sum_{k=1}^{R} \frac{\sqrt{\mu_k} \, \gamma_k}{\mu_k + \eta} \, \mathbf{u}_k \, .$$

Good news: Noise is attenuated

Bad news: Some components in measurement space are attenuated

Summary

- The Moore-Penrose pseudoinverse is an elegant theoretical construct
- It can be realized by SVD methods or iterative algorithms
- When applied to noisy data, it yields the MNLS solution
- MNLS solutions have no null functions but large noise
- Tikhonov and other regularizing methods can control noise, but at the expense of resolution