Representations of functions

(with application to the design of deformable mirrors)

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• Objects and wavefronts are functions, but our computer insists on discrete vectors. What is the "best" way to discretize a function?

• Where is the "best" place to put the actuators on a deformable mirror? (Question posed by Stéphane Chamot)

$\boxed{\text{OUTLINE}}$

- Review
  Scalar product, adjoints, outer products, ...
  Pseudoinverses, projection operators, ...

- Some useful differentiation formulas

- Object space, representation space and coefficient space

- Least-squares representations in orthonormal functions

- Least-squares representations in non-orthonormal functions

- Karhunen-Loève expansion

- Design of deformable mirrors

References

Barrett and Myers, FOUNDATIONS OF IMAGE SCIENCE

Chap. 1, Vectors and Operators (for review)

Appendix A, Matrix Algebra (for differentiation formulas)

Chap. 7, Deterministic Models (for the meat of this lecture)

$\boxed{\text{Functions}}$

The functions of interest in this lecture will be real-valued, square-integrable functions of a 2D position vector $\mathbf{r} = (x, y)$ with support $\mathbf{S}$.

Notation:

General function: $f(\mathbf{r}) = f(x, y)\,, \qquad f(\mathbf{r}) = 0 \;\forall\, \mathbf{r} \notin \mathbf{S}$

Same function regarded as vector in Hilbert space $\mathbb{L}_2(\mathbf{S})$: $\mathbf{f}$

Norm: $||\mathbf{f}\,||$

Definition of scalar product and norm:

$$(\mathbf{f}_1, \mathbf{f}_2) = \int_{\mathbf{S}} d^2r\, f_1(\mathbf{r})\, f_2(\mathbf{r})\,, \qquad ||\mathbf{f}\,||^2 = (\mathbf{f}\,, \mathbf{f}\,) < \infty$$

Alternative notation for scalar product: $(\mathbf{f}_1, \mathbf{f}_2) = \mathbf{f}_1^\dagger \mathbf{f}_2$

$\boxed{\text{Operators}}$

Of interest in this lecture are *discretization operators*, denoted $\boldsymbol{\mathcal{D}}$.

Consider a set of 2D functions $\{\psi_n(\mathbf{r}), n = 1, ...N\}$. The operator $\boldsymbol{\mathcal{D}}_\psi$ acts on a function $f(\mathbf{r})$ and yields an ND vector with components

$$\alpha_n \equiv [\boldsymbol{\mathcal{D}}_\psi \, \mathbf{f}]_n = \int_{\mathbf{S}} d^2 r \, \psi_n(\mathbf{r}) \, f(\mathbf{r}) = \boldsymbol{\psi}_n^\dagger \mathbf{f} \; ,$$

or, in vector form,

$$\boldsymbol{\alpha} = \boldsymbol{\mathcal{D}}_\psi \, \mathbf{f} \, .$$

Adjoints

Recall the basic definition of adjoint. If operator $\boldsymbol{\mathcal{O}}$ maps $\mathbb{U} \to \mathbb{V}$, then $\boldsymbol{\mathcal{O}}^\dagger$ is defined by

$$(\mathbf{g}, \boldsymbol{\mathcal{O}}\mathbf{f})_{\mathbb{V}} = (\boldsymbol{\mathcal{O}}^\dagger \mathbf{g}, \mathbf{f})_{\mathbb{U}}$$

By this definition,

$$[\boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha}](\mathbf{r}) = \sum_{n=1}^{N} \alpha_n \, \psi_n(\mathbf{r}) \, .$$

(No complex conjugate is needed since all quantities are assumed real.)

Some useful differentiation formulas

Let $\mathbf{x}$ be an $N \times 1$ vector and $Q(\mathbf{x})$ be a real scalar-valued function. The gradient of $Q(\mathbf{x})$ is an $N \times 1$ vector with components given by

$$[\nabla_{\mathbf{x}} Q]_n = \frac{\partial}{\partial x_n} Q(\mathbf{x})$$

Specifically, if $\mathbf{a}$ is a real $N \times 1$ vector and $\mathbf{A}$ is an $N \times N$ Hermitian matrix (both independent of $\mathbf{x}$), then

$$\nabla_{\mathbf{x}}\, \mathbf{a}^t \mathbf{x} = \mathbf{a}\,, \qquad \nabla_{\mathbf{x}}\, \mathbf{x}^t \mathbf{A} \mathbf{x} = 2\mathbf{A}\mathbf{x}\,.$$

Note also that

$$\mathbf{x}^t \mathbf{A} \mathbf{x} = \mathrm{tr}(\mathbf{A}\mathbf{x}\mathbf{x}^t)$$

For the complex case and many other formulas, see B&M Appendix A.

Pseudoinverses

From lecture 6, the Moore-Penrose pseudoinverse of an operator $\boldsymbol{\mathcal{H}}$ is the unique operator $\boldsymbol{\mathcal{H}}^+$ that satisfies the four Penrose equations:

$$\text{Penrose Eq. 1: } \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^+\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}\,. \tag{1.130a}$$

$$\text{Penrose Eq. 2: } \boldsymbol{\mathcal{H}}^+\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^+ = \boldsymbol{\mathcal{H}}^+\,. \tag{1.130b}$$

$$\text{Penrose Eq. 3: } (\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^+)^\dagger = \boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^+\,. \tag{1.130c}$$

$$\text{Penrose Eq. 4: } (\boldsymbol{\mathcal{H}}^+\boldsymbol{\mathcal{H}})^\dagger = \boldsymbol{\mathcal{H}}^+\boldsymbol{\mathcal{H}}\,. \tag{1.130d}$$

If $\boldsymbol{\mathcal{H}}$ has a true inverse, it satisfies all four of the Penrose equations.

The MP pseudoinverse exists and is unique for all matrices, all operators with a finite-dimensional range (hence all CD operators) and many integral operators.

Useful identities (for reference)

$$\left[\mathcal{H}^{+}\right]^{+} = \mathcal{H}\,; \tag{1.147}$$

$$\mathcal{H}^{+} = \left(\mathcal{H}^{\dagger}\mathcal{H}\right)^{+}\mathcal{H}^{\dagger}\,; \tag{1.148}$$

$$\mathcal{H}^{+} = \mathcal{H}^{\dagger}\left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+}\,; \tag{1.149}$$

$$\left[\mathcal{H}^{\dagger}\right]^{+} = \left[\mathcal{H}^{+}\right]^{\dagger}\,; \tag{1.150}$$

$$\left[\mathcal{H}^{\dagger}\right]^{+} = \left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+}\mathcal{H}\,; \tag{1.151}$$

$$\left[\mathcal{H}^{+}\mathcal{H}\right]^{+} = \mathcal{H}^{+}\mathcal{H}\,; \tag{1.152}$$

$$\mathcal{H}^{+}\mathcal{H}\mathcal{H}^{\dagger} = \mathcal{H}^{\dagger}\,; \tag{1.153}$$

$$\mathcal{H}^{\dagger}\mathcal{H}\mathcal{H}^{+} = \mathcal{H}^{\dagger}\,; \tag{1.154}$$

$$\mathcal{H}^{+}\mathcal{H} = \left(\mathcal{H}^{\dagger}\mathcal{H}\right)^{+}\left(\mathcal{H}^{\dagger}\mathcal{H}\right) = \left(\mathcal{H}^{\dagger}\mathcal{H}\right)\left(\mathcal{H}^{\dagger}\mathcal{H}\right)^{+}\;; \tag{1.155}$$

$$\mathcal{H}\mathcal{H}^{+} = \left(\mathcal{H}\mathcal{H}^{\dagger}\right)^{+}\mathcal{H}\mathcal{H}^{\dagger} = \mathcal{H}\mathcal{H}^{\dagger}\left(\mathcal{H}\mathcal{H}^{\dagger}\right)^{+}\;; \tag{1.156}$$

$$\left[\mathcal{H}^{\dagger}\mathcal{H}\right]^{+} = \mathcal{H}^{+}\left[\mathcal{H}^{\dagger}\right]^{+}\;; \tag{1.157}$$

$$\left[\mathcal{H}^{\dagger}\mathcal{H}\right]^{+} = \mathcal{H}^{+}\left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+}\mathcal{H} = \mathcal{H}^{\dagger}\left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+}\left[\mathcal{H}^{\dagger}\right]^{+}\;; \tag{1.158}$$

$$\left[\mathcal{H}\mathcal{H}^{\dagger}\right]^{+} = \left[\mathcal{H}^{\dagger}\right]^{+}\mathcal{H}^{+}\,. \tag{1.159}$$

All of these identities can be proved either directly from the Penrose equations or from the SVD representations of $\mathcal{H}$ and $\mathcal{H}^{+}$.

$\boxed{\text{Subspaces}}$

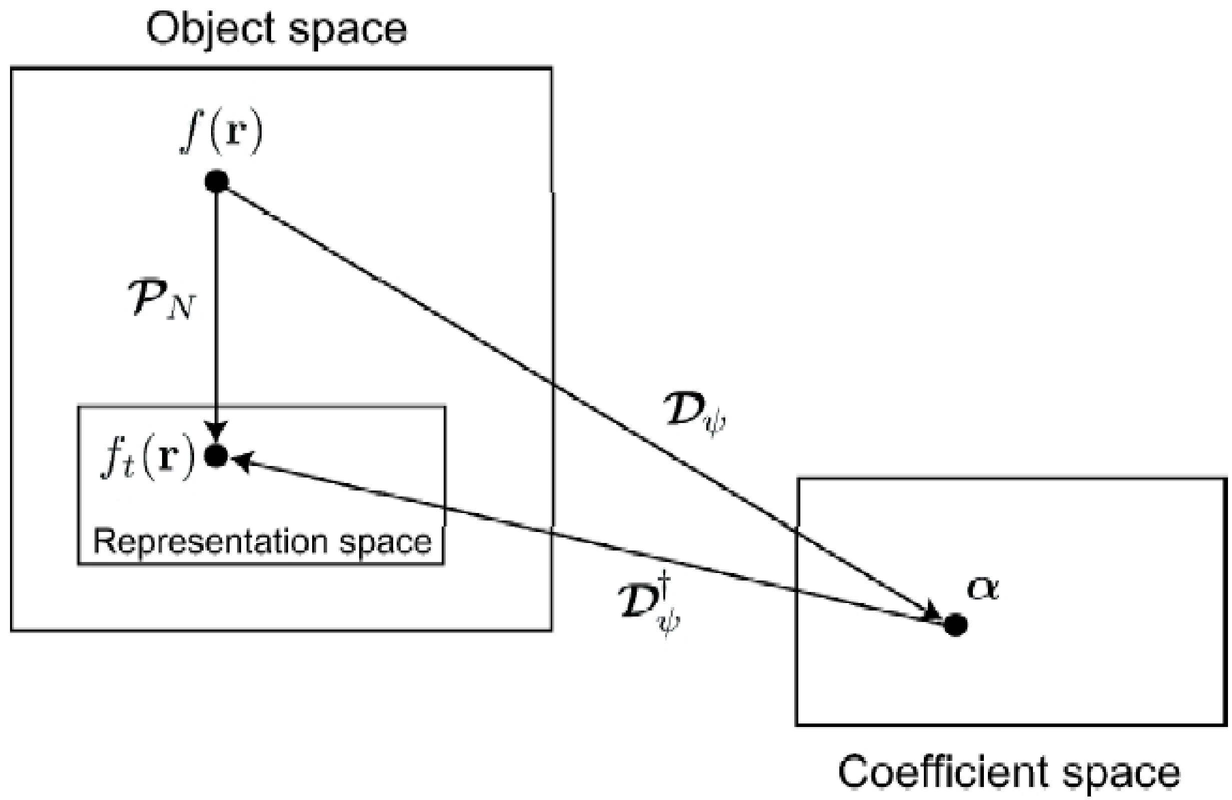
As above, consider a set of functions $\{\psi_n(\mathbf{r}), n = 1, ...N\}$ and the associated discretization operator $\boldsymbol{\mathcal{D}}_\psi$.

If the functions are square-integrable over support $\mathbf{S}$ and linearly independent, they form a basis (not necessarily an orthonormal one) for an N-dimensional subspace of $\mathbb{L}_2(\mathbf{S})$ called *representation space*.

An arbitrary function in representation space can be written as

$$f_t(\mathbf{r}) = \sum_{n=1}^{N} \alpha_n \, \psi_n(\mathbf{r}) \, , \qquad \mathbf{f}_t = \boldsymbol{\mathcal{D}}_\psi^\dagger \, \boldsymbol{\alpha} \, .$$

Note that $\boldsymbol{\alpha}$ itself, though it is N-dimensional, is not in representation space since is it not made up of functions. Instead we define a space isomorphic to representation space and call it *coefficient space*.



Barrett and Myers, Fig. 7.2

Important points for this diagram:

Representation space is subspace of object space

Coefficient space is isomorphic to representation space

Diagram is for orthonormal functions

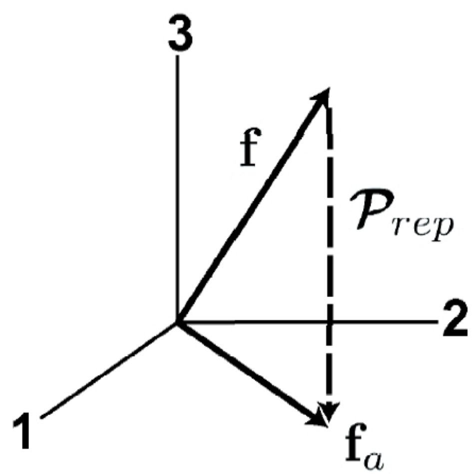
(Ignore the mappings $\mathcal{P}_N$ and $\mathcal{D}_\psi$ for now.)

Projection onto representation space

From lecture 7 we know that the orthogonal projector onto the ND representation space is given by

$$\mathcal{P}_{rep}\,\mathrm{f} = \mathcal{D}_{\psi}^{+}\mathcal{D}_{\psi}\mathrm{f}\,.$$

This operator yields the closest point in representation space to an arbitrary object f, i.e., a least-squares fit:



The 1-2 plane is representation space in this figure

Orthonormal expansion functions

So far we have assumed that the functions $\{\psi_n(\mathbf{r}), n = 1, ...N\}$ are linearly independent and square-integrable; now let us assume also that they are orthonormal:

$$\boldsymbol{\psi}_n^\dagger \boldsymbol{\psi}_m = \int_{\mathbf{S}} d^2r\, \psi_n(\mathbf{r})\, \psi_m(\mathbf{r}) = \delta_{nm}$$

Since the functions are also linearly independent, they form a complete set in representation space.

From orthonormality and completeness, can show that

$$\boldsymbol{\mathcal{D}}_\psi^+ = \boldsymbol{\mathcal{D}}_\psi^\dagger$$

Proof: Plug into the Penrose equations or use SVD

Orthonormal expansion functions – cont.

From last page:

$$\boldsymbol{\mathcal{D}}_\psi^+ = \boldsymbol{\mathcal{D}}_\psi^\dagger$$

Thus, for orthonormal functions,

$$\boldsymbol{\mathcal{P}}_{rep}\, \mathbf{f} = \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\mathcal{D}}_\psi\, \mathbf{f}\,,$$

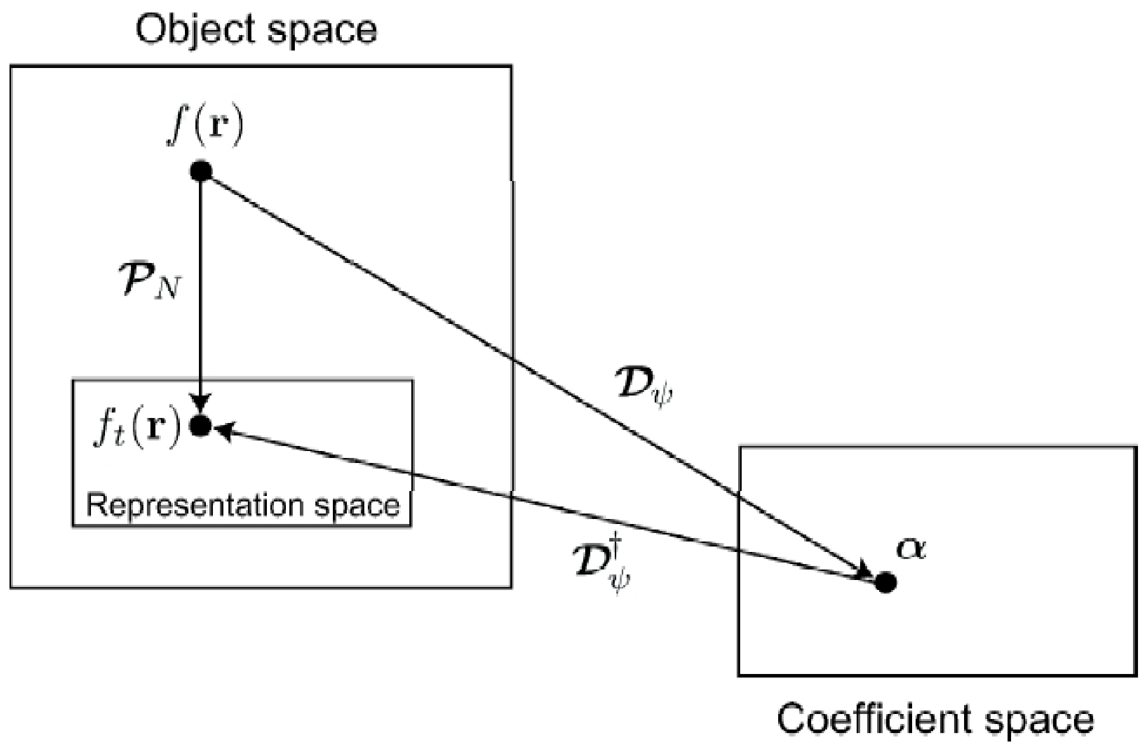
Interpretation:

1. Compute coefficients by scalar products, $\alpha_n = \boldsymbol{\psi}_n^\dagger \mathbf{f}$
2. Use the coefficients as weights: $[\boldsymbol{\mathcal{P}}_{rep}\, \mathbf{f}](\mathbf{r}) = \sum_{n=1}^N \alpha_n\, \psi_n(\mathbf{r})$

Result is least-squares fit of $\sum_{n=1}^N \alpha_n\, \psi_n(\mathbf{r})$ to $f(\mathbf{r})$

Fig. 7.2 revisited

(Slight change of notation: $\mathcal{P}_{rep}$ called $\mathcal{P}_N$ here)



Main point: To get from $\mathbf{f}$ to *nearest* point in representation space with orthonormal functions, apply operator $\mathcal{D}_\psi^\dagger \mathcal{D}_\psi$, which is easy (no inverses required).

Brute-force minimization of squared error

Now suppose we are given a function $f(\mathbf{r})$ and a set of expansion functions $\{\psi_n(\mathbf{r}), n = 1, ...N\}$, which are not necessarily orthonormal.

We want to minimize the squared error:

$$||\mathbf{f} - \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha}||^2 = \int_{\mathbf{S}} d^2r \left[f(\mathbf{r}) - \sum_{n=1}^{N} \alpha_n \psi_n(\mathbf{r}) \right]^2 = \text{ minimum} \,,$$

by choice of $\{\alpha_n\}$.

Apply the differentiation formulas given earlier:

$$\nabla_{\boldsymbol{\alpha}} ||\mathbf{f} - \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha}||^2 = \nabla_{\boldsymbol{\alpha}} \left[||\mathbf{f}||^2 - 2\mathbf{f}^\dagger \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha} + ||\boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha}||^2 \right]$$

$$= \nabla_{\boldsymbol{\alpha}} \left[-2(\boldsymbol{\mathcal{D}}_\psi \mathbf{f})^\dagger \boldsymbol{\alpha} + \boldsymbol{\alpha}^\dagger \boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha} \right] = -2\boldsymbol{\mathcal{D}}_\psi \mathbf{f} + 2\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha} = 0$$

Brute-force minimization – cont.

From last slide, the best coefficients (in the LS sense) must satisfy

$$\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha} = \boldsymbol{\mathcal{D}}_\psi \, \mathbf{f} \, .$$

For orthonormal functions, $\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger$ is the $N \times N$ unit matrix, so

$$\boldsymbol{\alpha} = \boldsymbol{\mathcal{D}}_\psi \mathbf{f}$$

For non-orthonormal functions, the general solution is

$$\boldsymbol{\alpha} = \left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger \right]^+ \boldsymbol{\mathcal{D}}_\psi \, \mathbf{f} \, ,$$

where $\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger$ is an $N \times N$ matrix with elements

$$\left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger \right]_{nm} = \int_{\mathbf{S}} d^2 r \, \psi_n(\mathbf{r}) \, \psi_m(\mathbf{r}) \, .$$

If the functions are linearly independent, the true inverse exists and

$$\boldsymbol{\alpha} = \left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger \right]^{-1} \boldsymbol{\mathcal{D}}_\psi \, \mathbf{f} \, ,$$

Brute-force minimization – cont.

From last slide, the general solution is

$$\boldsymbol{\alpha} = \left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger\right]^+ \boldsymbol{\mathcal{D}}_\psi \, \mathrm{f}\,,$$

Coefficients that satisfy this equation will be used to construct a representation given by

$$\mathrm{f}_{rep} = \boldsymbol{\mathcal{D}}_\psi^\dagger \, \boldsymbol{\alpha} = \boldsymbol{\mathcal{D}}_\psi^\dagger \, \left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger\right]^+ \boldsymbol{\mathcal{D}}_\psi \, \mathrm{f}\,.$$

By use of identity (1.149), we get

$$\mathrm{f}_{rep} = \boldsymbol{\mathcal{D}}_\psi^+ \boldsymbol{\mathcal{D}}_\psi \, \mathrm{f} \, = \boldsymbol{\mathcal{P}}_{rep} \, \mathrm{f} \, .$$

Thus the formula at the top of this page does indeed result in the (orthogonal) projection of f onto representation space, i.e., a least-squares fit.

Summary so far

$$\boldsymbol{\alpha} = [\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger]^+ \boldsymbol{\mathcal{D}}_\psi \, \mathbf{f}$$

• Given a set of expansion functions, these coefficients yield the representation with smallest squared error for every object $\mathbf{f}$.

• If the expansion functions are orthonormal, application of the formula is trivial.

• If the expansion functions are linearly independent but not orthogonal, a matrix inverse is required.

• The inversion step is equivalent to orthonormalizing the expansion functions

Remaining question:

- How do we choose the expansion functions in the first place?

How to choose the expansion functions?

Answer to this question is necessarily statistical, involving an ensemble of objects $\mathbf{f}$.

For this ensemble, one might:

- Minimize the ensemble mean-squared error
- Make the coefficients uncorrelated
- Try to make the coefficients statistically independent
- Maximize task performance

Ensemble mean-squared error

Criterion:

$$\text{EMSE} = \left\langle ||\mathbf{f} - \boldsymbol{\mathcal{D}}_{\psi}^{\dagger}\boldsymbol{\alpha}||^2 \right\rangle = \text{ minimum} .$$

Now a double minimization: Choice of $\{\alpha_n\}$ for given $\{\boldsymbol{\psi}_n\}$, then choice of $\{\boldsymbol{\psi}_n\}$.

Restrict attention to orthonormal functions. Then one can show (B&M, Sec. 7.1.4) that the first minimization yields

$$\text{EMSE} = \left\langle ||\mathbf{f}\,||^2 \right\rangle - \left\langle ||\boldsymbol{\alpha}||^2 \right\rangle \; .$$

Thus the second minimization is equivalent to *maximizing* $\langle ||\boldsymbol{\alpha}||^2 \rangle$. Do this by choosing the first N eigenfunctions (N largest eigenvalues) of the autocorrelation operator as the expansion functions.

This is the *Karhunen-Loève expansion*.

Application to the design of deformable mirrors

Problem posed by Stéphane Chamot:

Suppose a pupil function is represented accurately by an expansion in K Zernike polynomials and that the mean vector and the covariance matrix for the coefficients are known.

Suppose you want to match this pupil function as accurately as possible (in an EMSE sense) with a deformable mirror with N actuators.

Where do you put the actuators?

New wrinkles:

Object space (wavefront space) is finite-dimensional

Expansion functions (influence functions) are not orthonormal

Interest is in choice of functions, not coefficients

Assumptions

- For some K, the wavefront is given to acceptable accuracy by

$$f_{wf}(\mathbf{r}) = \sum_{k=1}^{K} \beta_k \, Z_k(\mathbf{r}) \,, \qquad \mathbf{f}_{wf} = \boldsymbol{\mathcal{D}}_Z^{\dagger} \boldsymbol{\beta}$$

  where $Z_k(\mathbf{r})$ is the k^{th} Zernike polynomial.

- The autocorrelation matrix of the coefficients is known:

$$[\mathbf{R}_{\boldsymbol{\beta}}]_{kk'} = \langle \beta_k \, \beta_{k'} \rangle \ , \qquad \mathbf{R}_{\boldsymbol{\beta}} = \left\langle \boldsymbol{\beta} \, \boldsymbol{\beta}^{\dagger} \right\rangle \ .$$

- Mirror influence functions $\psi_n(\mathbf{r})$ are identical except for shift:

$$\psi_n(\mathbf{r}) = \psi(\mathbf{r} - \mathbf{r}_n) \,, \qquad n = 1, ..., N$$

- Wavefront sensor is perfect – no noise in measurement of $\boldsymbol{\beta}$.

MSE for a fixed wavefront

The MSE, before ensemble averaging, is given by

$$\mathsf{MSE} = ||\mathbf{f}_{wf} - \mathbf{f}_{dm}||^2 = ||\boldsymbol{\mathcal{D}}_Z^\dagger \boldsymbol{\beta} - \boldsymbol{\mathcal{D}}_\psi^\dagger \boldsymbol{\alpha}||^2 \,.$$

From above, we know that the optimum choice of $\boldsymbol{\alpha}$ for a given $\boldsymbol{\beta}$ is

$$\boldsymbol{\alpha} = \left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger\right]^+ \boldsymbol{\mathcal{D}}_\psi \,\mathbf{f} = \left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger\right]^+ \boldsymbol{\mathcal{D}}_\psi \,\boldsymbol{\mathcal{D}}_Z^\dagger \boldsymbol{\beta} \,.$$

Hence, with (1.149) again,

$$\mathbf{f}_{dm} = \boldsymbol{\mathcal{D}}_\psi^\dagger \,\boldsymbol{\alpha} = \boldsymbol{\mathcal{D}}_\psi^\dagger \,\left[\boldsymbol{\mathcal{D}}_\psi \boldsymbol{\mathcal{D}}_\psi^\dagger\right]^+ \boldsymbol{\mathcal{D}}_\psi \,\boldsymbol{\mathcal{D}}_Z^\dagger \boldsymbol{\beta} = \boldsymbol{\mathcal{P}}_{dm} \,\boldsymbol{\mathcal{D}}_Z^\dagger \boldsymbol{\beta} \,,$$

where $\boldsymbol{\mathcal{P}}_{dm}$ is the projector onto "deformable-mirror space":

$$\boldsymbol{\mathcal{P}}_{dm} = \boldsymbol{\mathcal{D}}_\psi^+ \,\boldsymbol{\mathcal{D}}_\psi \,.$$

The MSE is thus

$$\mathsf{MSE} = ||(\mathbf{I} - \boldsymbol{\mathcal{P}}_{dm}) \,\boldsymbol{\mathcal{D}}_Z^\dagger \boldsymbol{\beta}||^2 \,.$$

Ensemble averaging

From last slide:

$$\text{MSE} = ||(\mathbf{I} - \boldsymbol{\mathcal{P}}_{dm})\,\boldsymbol{\mathcal{D}}_Z^\dagger \boldsymbol{\beta}||^2 \,.$$

Define a new operator $\boldsymbol{\mathcal{O}}$ by

$$\boldsymbol{\mathcal{O}}^\dagger = (\mathbf{I} - \boldsymbol{\mathcal{P}}_{dm})\,\boldsymbol{\mathcal{D}}_Z^\dagger$$

so that

$$\text{MSE} = ||(\mathbf{I} - \boldsymbol{\mathcal{P}}_{dm})\,\boldsymbol{\mathcal{D}}_Z^\dagger \boldsymbol{\beta}||^2 \equiv ||\boldsymbol{\mathcal{O}}^\dagger \boldsymbol{\beta}||^2 = [\boldsymbol{\mathcal{O}}^\dagger \boldsymbol{\beta}]^\dagger \, [\boldsymbol{\mathcal{O}}^\dagger \boldsymbol{\beta}] = \boldsymbol{\beta}^\dagger \boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^\dagger \, \boldsymbol{\beta} \,.$$

Now do the average:

$$\text{MSE} = \left\langle \boldsymbol{\beta}^\dagger \boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^\dagger \, \boldsymbol{\beta} \right\rangle = \text{tr}\, \left\{ \boldsymbol{\mathcal{O}}^\dagger \left\langle \boldsymbol{\beta}\,\boldsymbol{\beta}^\dagger \right\rangle \boldsymbol{\mathcal{O}} \right\} = \text{tr}\, \left\{ \boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^\dagger \mathbf{R}_{\boldsymbol{\beta}} \right\} \,.$$

Note that $\boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^\dagger$ is a $K \times K$ matrix:

$$\boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^\dagger = \boldsymbol{\mathcal{D}}_Z\,(\mathbf{I} - \boldsymbol{\mathcal{P}}_{dm})\,\boldsymbol{\mathcal{D}}_Z^\dagger = \mathbf{I} - \boldsymbol{\mathcal{D}}_Z\,\boldsymbol{\mathcal{P}}_{dm}\,\boldsymbol{\mathcal{D}}_Z^\dagger \,.$$

Detail on the matrix calculation

$$\left[\boldsymbol{\mathcal{O}}\,\boldsymbol{\mathcal{O}}^{\dagger}\right]_{kk'} = \delta_{kk'} - \left[\boldsymbol{\mathcal{D}}_Z\,\boldsymbol{\mathcal{P}}_{dm}\,\boldsymbol{\mathcal{D}}_Z^{\dagger}\right]_{kk'}\,.$$

Now use (1.149) in reverse:

$$\boldsymbol{\mathcal{D}}_Z\,\boldsymbol{\mathcal{P}}_{dm}\,\boldsymbol{\mathcal{D}}_Z^{\dagger} = \boldsymbol{\mathcal{D}}_Z\,\boldsymbol{\mathcal{D}}_\psi^{\dagger}\,\left[\boldsymbol{\mathcal{D}}_\psi\,\boldsymbol{\mathcal{D}}_\psi^{\dagger}\right]^{+}\boldsymbol{\mathcal{D}}_\psi\,\boldsymbol{\mathcal{D}}_Z^{\dagger}$$

$$\left[\boldsymbol{\mathcal{D}}_Z\,\boldsymbol{\mathcal{P}}_{dm}\,\boldsymbol{\mathcal{D}}_Z^{\dagger}\right]_{kk'} = \sum_{n=1}^{N}\sum_{n'=1}^{N}\left[\boldsymbol{\mathcal{D}}_\psi\,\boldsymbol{\mathcal{D}}_Z^{\dagger}\right]_{nk}\left[\left(\boldsymbol{\mathcal{D}}_\psi\,\boldsymbol{\mathcal{D}}_\psi^{\dagger}\right)^{+}\right]_{nn'}\left[\boldsymbol{\mathcal{D}}_\psi\,\boldsymbol{\mathcal{D}}_Z^{\dagger}\right]_{n'k'}$$

where

$$\left[\boldsymbol{\mathcal{D}}_\psi\,\boldsymbol{\mathcal{D}}_Z^{\dagger}\right]_{nk} = \int_{\mathbf{S}} d^2r\,\psi_n(\mathbf{r})\,Z_k(\mathbf{r})$$

A strategy for optimizing the mirror (per SC)

$$\text{EMSE} = \text{tr}\left\{\boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^{\dagger}\mathbf{R}_{\boldsymbol{\beta}}\right\}, \qquad \boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^{\dagger} = \mathbf{I} - \boldsymbol{\mathcal{D}}_Z\,\boldsymbol{\mathcal{P}}_{dm}\,\boldsymbol{\mathcal{D}}_Z^{\dagger}\,.$$

1. Collect a large set of actual wavefronts, do Zernike decompositions, and estimate $\mathbf{R}_{\boldsymbol{\beta}}$. Call the estimate $\hat{\mathbf{R}}_{\boldsymbol{\beta}}$.

2. For some set of actuator positions, estimate EMSE by tr $\left\{\boldsymbol{\mathcal{O}}\boldsymbol{\mathcal{O}}^{\dagger}\hat{\mathbf{R}}_{\boldsymbol{\beta}}\right\}$

3. Perturb the actuator configuration, estimate new EMSE

4. Accept or reject the new configuration by simulated-annealing rule:

$$\text{Pr(acc)} = \min\{1, \exp[-\Delta\text{EMSE}/kT]\}$$

5. Iterate

6. Anneal ($kT \rightarrow 0$)

Relation to system alignment

Richard Paxman et al., JOSA A, 1985

SVD allows a unique decomposion of object space into two subspaces: null space and measurement space. Measurement space corresponds to nonzero eigenvalues of the $\mathcal{H}^\dagger\mathcal{H}$ operator and thus contains the part of the object that can be imaged by a given system.

The Karhunen-Loève expansion allows a second decomposition of object space, with subspace we call *interest space* and *indifference space*. Interest space is spanned by eigenfunctions of the autocorrelation operator $\mathcal{R}_{\mathbf{f}}$ with significantly nonzero eigenvalues. It contains the bulk of the "energy" for the object class described by $\mathcal{R}_{\mathbf{f}}$.

System design for minimum EMSE is equivalent to maximizing the overlap between measurement space and interest space.

System alignment – details

SVD – expand object in eigenfunctions of $\boldsymbol{\mathcal{H}}^\dagger\boldsymbol{\mathcal{H}}$:

$$\mathbf{f} = \sum_{n=1}^{N} \alpha_n \mathbf{u}_n = \sum_{n=1}^{R} \alpha_n \mathbf{u}_n + \sum_{n=R+1}^{N} \alpha_n \mathbf{u}_n = \mathbf{f}_{meas} + \mathbf{f}_{null}\,,$$

$$\boldsymbol{\mathcal{H}}^\dagger\boldsymbol{\mathcal{H}}\mathbf{u}_n = \mu_n\,\mathbf{u}_n\,, \qquad \mu_n \approx 0 \text{ for } n > R\,.$$

KL – expand object in eigenfunctions of autocorrelation operator:

$$\mathbf{f} = \sum_{m=1}^{N} \gamma_m \mathbf{w}_m = \sum_{m=1}^{R'} \gamma_m \mathbf{w}_m + \sum_{m=R'+1}^{N} \gamma_m \mathbf{w}_m = \mathbf{f}_{int} + \mathbf{f}_{indiff}\,,$$

$$\boldsymbol{\mathcal{R}}\mathbf{w}_m = \lambda_m\,\mathbf{w}_m\,, \qquad \lambda_m \approx 0 \text{ for } m > R'\,.$$

Summary

- Linear representations are superpositions of expansion functions
- Pseudoinverses allow optimum choice of coefficients
- Optimum expansion functions for EMSE are KL functions
- Simulated annealing can be used to design system for min. EMSE
- Min. EMSE $\Leftrightarrow$ "aligning" measurement space and interest space.
- For DMs, KL uses correlation matrix of Zernikes
- In DM problem, measurement space spanned by influence fcns., interest space by significant Zernikes