

Lecture 15

Theory of random processes
Part III: Poisson random processes

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OUTLINE

- Poisson and independence
- Poisson and rarity; binomial selection
- Poisson point processes: definition and basic properties
- Mean and covariance (stationary and nonstationary)
- Doubly stochastic Poisson variables and processes
- Poisson and non-Poisson light sources
- Karhunen-Loève for the Poisson case
- Characteristic functionals

Poisson and independence

The postulates that lead to the Poisson law are:

(a) The number of counts in any time interval is statistically independent of the number in any other *nonoverlapping* interval

(b) If we consider a very small time interval of duration ΔT , the probability of a single count in this interval approaches some constant a times ΔT , *i.e.*,

$$\text{Pr}(1 \text{ in } \Delta T) = a\Delta T, \quad (11.1)$$

(Note that a has dimensions of reciprocal time, will soon be interpreted as a rate.)

(c) The probability of more than one count in a small interval ΔT is zero:

$$\text{Pr}(1 \text{ in } \Delta T) + \text{Pr}(0 \text{ in } \Delta T) = 1. \quad (11.2)$$

THE POISSON PROBABILITY LAW

If the postulates are satisfied with $a = \text{const.}$, then

$$\Pr(N \text{ in } T) = \frac{(aT)^N}{N!} \exp(-aT). \quad (11.13)$$

(Derivation in B&M)

Mean = rate \times time:

$$\overline{N} = aT$$

Variance = mean:

$$\text{Var}(N) = \langle (N - \overline{N})^2 \rangle = aT = \overline{N}$$

A generalization

Assume postulates are satisfied but rate is a *nonrandom* function of time, $a(t)$.

Example: $a(t)$ could be a short pulse, provided pulse shape and amplitude are always the same on each repetition.

Then

$$\Pr[N \text{ in } (t, t + T)] = \frac{\bar{N}^N}{N!} \exp(-\bar{N}), \quad (11.18)$$

where now \bar{N} is a function of time given by

$$\bar{N} = \int_t^{t+T} dt' a(t'). \quad (11.19)$$

Variance still equals mean:

$$\text{Var}(N) = \langle (N - \bar{N})^2 \rangle = \int_t^{t+T} dt' a(t') = \bar{N}$$

Punchline:

Variation in rate does not spoil independence/Poissonicity provided rate is not random.

Poisson and rarity

Consider a set of M Bernoulli trials (coin flips, say), where the probability of success (heads) is p . The total number N of successes in M trials is given by the binomial law:

$$\Pr(N|M, p) = \binom{M}{n} p^n q^{M-n}$$

Mean number of successes: $\bar{N} = Mp$.

Now let M become very large and p become very small in such a way that Mp (or \bar{N}) remains constant.

$$\lim_{M \rightarrow \infty} \Pr(N|M, p) = \frac{(\bar{N})^N}{N!} \exp(-\bar{N}) . \quad (11.21)$$

“Rarity begets Poissonicity”

Binomial selection of a Poisson

In many applications of the binomial law, the number of trials M is random.

Important example: radiation detection by an inefficient photon-counting detector.

Suppose M photons are incident on the detector in time T and that each has a probability η (called the *quantum efficiency*) of producing a photoelectron. If the photoelectric interactions are statistically independent, then the *conditional* probability $\Pr(N|M)$ for getting N photoelectrons is a binomial. The marginal probability $\Pr(N)$, however, is given by

$$\Pr(N) = \sum_{M=N}^{\infty} \Pr(N|M) \Pr(M). \quad (11.22)$$

Binomial selection – cont.

Now suppose that the photons satisfy the three postulates so that M is a Poisson random variable. With $\Pr(N|M)$ being binomial, we have

$$\begin{aligned}\Pr(N) &= \sum_{M=N}^{\infty} \Pr(N|M) \Pr(M) \\ &= \sum_{M=N}^{\infty} \binom{M}{N} \eta^N (1-\eta)^{M-N} \exp(-\bar{M}) \frac{\bar{M}^M}{M!}.\end{aligned}\tag{11.23}$$

A change of variables, $K = M - N$, and a little algebra shows that

$$\Pr(N) = \exp(-\eta\bar{M}) \frac{(\eta\bar{M})^N}{N!}.\tag{11.24}$$

Thus N obeys a Poisson law with mean $\bar{N} = \eta\bar{M}$.

Binomial selection theorem

Binomial selection of a Poisson yields a Poisson, and the mean of the output of the selection process is the mean of the input times the binomial probability of success.

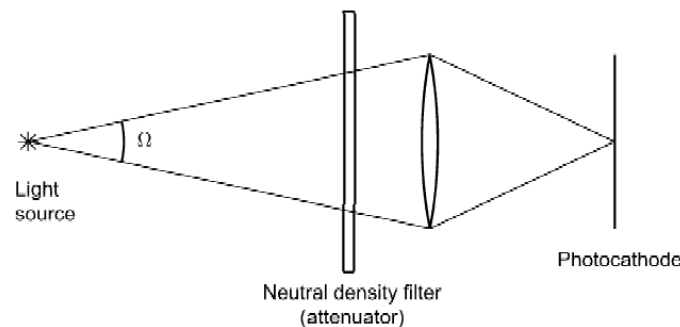
Interesting corollary: Number of times that the photon is *not* detected is also a Poisson random variable.

Moreover, the number of nondetections is statistically independent of the number of detections.

In a sequence of Bernoulli trials where the number of attempts is random, the number of successes is independent of the number of failures if and only if the number of attempts is a Poisson random variable.

Cascaded binomial selection

Suppose a light source emits photons randomly in all directions. Some of the photons fall on the photocathode of a photomultiplier tube and some of these produce photoelectrons.



Assume:

- Photon emissions are independent, mean \overline{M}
- Photon directions are independent, isotropic
- Fraction $\Omega/4\pi$ of photons fall on PMT
- Photoelectrons are produced independently
- Fraction η of photons produce photoelectrons

Cascaded binomial selection – cont.

Since binomial selection of a Poisson yields a Poisson, and a Poisson is fully characterized by its mean, the probability law on the number of photoelectrons is

$$\Pr(N) = \frac{(\overline{N})^N}{N!} \exp(-\overline{N}),$$

where the mean number of photoelectrons given by

$$\overline{N} = \eta \frac{\Omega}{4\pi} \overline{M}.$$

Can continue this process, include other binomial selections (absorption, reflection, etc.)

Key Point: Poisson source followed by any number of binomial selections gives a Poisson.

Doubly stochastic Poisson random variables

Consider Poisson random variables where the Poisson mean is random.

Example: Fluctuating optical source

Though N is a discrete random variable with only integer values, its mean \bar{N} can take on any value in $(0, \infty)$. Thus random \bar{N} must be described by a probability density function $\text{pr}(\bar{N})$.

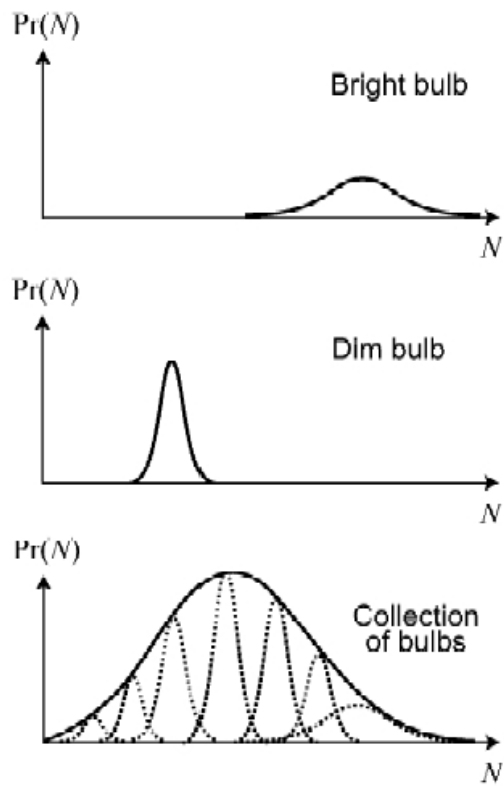
Mean number of photoelectrons now given by

$$\begin{aligned}\text{Pr}(N) &= \int_0^\infty d\bar{N} \text{Pr}(N|\bar{N}) \text{pr}(\bar{N}) \\ &= \frac{1}{N!} \int_0^\infty d\bar{N} \bar{N}^N \exp(-\bar{N}) \text{pr}(\bar{N}).\end{aligned}\tag{11.25}$$

This expression is called the *Poisson transform* of $\text{pr}(\bar{N})$

Example of interest in optics: Poisson transform of an exponential is a Bose-Einstein distribution

Illustration of doubly stochastic process



Doubly stochastic mean and variance

Mean number of photoelectrons is given by

$$\begin{aligned}\langle N \rangle &= \sum_{N=0}^{\infty} N \Pr(N) = \int_0^{\infty} d\bar{N} \left[\sum_{N=0}^{\infty} \frac{N}{N!} \bar{N}^N \exp(-\bar{N}) \right] \text{pr}(\bar{N}) \\ &= \int_0^{\infty} d\bar{N} \bar{N} \text{pr}(\bar{N}) \equiv \overline{\bar{N}}.\end{aligned}\tag{11.28}$$

The double-overbar indicates two separate averages: one with respect to $\Pr(N|\bar{N})$, another with respect to $\text{pr}(\bar{N})$.

Variance computed similarly (see B&M):

$$\text{Var}(N) = \langle (N - \overline{\bar{N}})^2 \rangle = \overline{\bar{N}} + \text{Var}(\bar{N}).\tag{11.32}$$

First term is Poisson variance appropriate to the average value of the mean. Second term, often called the *excess variance*, is the result of randomness in the Poisson mean.

Binomial selection from a non-Poisson source

Consider binomial selection (probability η) of a doubly stochastic Poisson random variable (mean $\overline{\overline{M}}$).

Mean:

$$\overline{\overline{N}} = \eta \overline{\overline{M}}$$

Variance:

$$\text{Var}(N) = \eta \overline{\overline{M}} + \eta^2 \text{Var}(\overline{\overline{M}}). \quad (11.39)$$

Note that Poisson part of the variance scales as η while the excess variance scales as η^2 .

Key point: Small efficiency reduces the excess variance relative to the Poisson part.

Again, rarity begets Poissonicity

So when is a light source Poisson?

- When its intensity is constant
- When intensity fluctuates but a small fraction of the light is used
- When its intensity fluctuates rapidly (temporal incoherence)
- When it is perfect monochromatic (spatially and temporally coherent)
- When it is in a quantum mechanical coherent state

And when is a light source not Poisson?

- When its intensity fluctuates slowly
- When it is one “mode” of a thermal radiation field
- When it is “quasithermal”
- When one considers an ensemble of sources

Multivariate Poisson statistics – quick reminder

Consider an array of photon-counting detectors, with g_j counts in j^{th} detector.

Data thus consist of set $\{g_j, j = 1, 2, \dots, J\}$

If individual counts g_j are Poisson (i.e. independent), then multivariate probability law is

$$\Pr(\{g_j\}) = \prod_{j=1}^J \Pr(g_j) = \prod_{j=1}^J \exp(-\bar{g}_j) \frac{\bar{g}_j^{g_j}}{g_j!}. \quad (11.40)$$

Covariance matrix

Independent counts are necessarily uncorrelated, so

$$K_{jk} = \langle [g_j - \bar{g}_j][g_k - \bar{g}_k] \rangle = \bar{g}_j \delta_{jk}. \quad (11.41)$$

Alternate notation:

Data vector \mathbf{g} has components $\{g_1, g_2, \dots, g_J\}$.

Mean vector $\bar{\mathbf{g}}$ has components $\{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_J\}$.

$$[\mathbf{K}_g] = \langle [\mathbf{g} - \bar{\mathbf{g}}] [\mathbf{g} - \bar{\mathbf{g}}]^t \rangle = \text{diag}(\bar{\mathbf{g}})$$

Temporal point processes

Output current of a photon-counting detector:

$$i(t) = \sum_{n=1}^N i_0(t - t_n), \quad (11.62)$$

where $i_0(t)$ is the current pulse produced by a count at $t = 0$, t_n is the time of occurrence of the n^{th} count ($0 < t_n \leq T$), and N is the total number of counts in $(0, T)$.

This current waveform can be written as

$$i(t) = z(t) * i_0(t), \quad (11.63)$$

where $z(t)$ is a random point process defined by

$$z(t) = \sum_{n=1}^N \delta(t - t_n), \quad (0 < t \leq T). \quad (11.64)$$

Objective: Understand statistical properties of $z(t)$

The stationary Poisson case

If the Poisson postulates hold and rate $a = \text{constant}$, then

$$\Pr(N \text{ in } T) = \frac{(aT)^N}{N!} \exp(-aT). \quad (11.13)$$

$$\text{pr}(t_n) = \frac{a}{N} = \frac{1}{T}$$

$$\langle z(t) \rangle = a$$

$$K_z(t, t') = \langle [z(t) - a][z(t') - a] \rangle = a \delta(t - t')$$

$$\text{Var}\{z(t)\} = \infty$$

$$S_{\Delta z}(\nu) = a$$

Key point:

All statistical properties of $z(t)$ determined by rate a .

The nonstationary Poisson case

If the Poisson postulates hold and rate $a(t)$ is a nonrandom function, then

$$\Pr(N \text{ in } T) = \frac{\left[\int_0^T dt' a(t') \right]^N}{N!} \exp \left[- \int_0^T dt' a(t') \right]$$
$$\text{pr}(t_n) = \frac{a(t_n)}{\int_0^T dt a(t)}, \quad (11.70)$$

$$\langle z(t) \rangle = a(t)$$

$$K_z(t, t') = a(t) \delta(t - t')$$

$$\text{Var}\{z(t)\} = \infty$$

Power spectral density not defined

Key point: *All* statistical properties of $z(t)$ determined by rate function $a(t)$.

Spectral description of a temporal Poisson process?

For stationary random processes, we use the Fourier domain since the autocorrelation operator is diagonal there. Recall from Lecture 13:

$$\langle F(\nu) F^*(\nu') \rangle = S(\nu) \delta(\nu - \nu').$$

Karhunen-Loève \Leftrightarrow Fourier domain

For nonstationary Poisson process, however, the autocorrelation operator is diagonal in the time domain. From last slide:

$$K_z(t, t') = a(t) \delta(t - t')$$

Fourier transformation would just *undisagonalize* it!

Karhunen-Loève \Leftrightarrow time domain

Spatial point processes

The spatial counterpart of the temporal point process $z(t)$ is $g(\mathbf{r})$, defined by

$$g(\mathbf{r}) = \sum_{n=1}^N \delta(\mathbf{r} - \mathbf{r}_n), \quad (11.72)$$

where \mathbf{r} is a spatial position vector.

For example, $g(\mathbf{r})$ could describe the pattern of photon interactions on a piece of film or (unbinned) output of an ideal photon-counting detector

Spatial Poisson postulates and their consequences

(a) The number of counts in any area A_1 is statistically independent of the number in any other nonoverlapping area A_2 .

(b) If we consider a very small area ΔA contained in A and centered on point \mathbf{r} , the probability of a single count in this area during an observation time approaches a deterministic function $b(\mathbf{r})$ times ΔA :

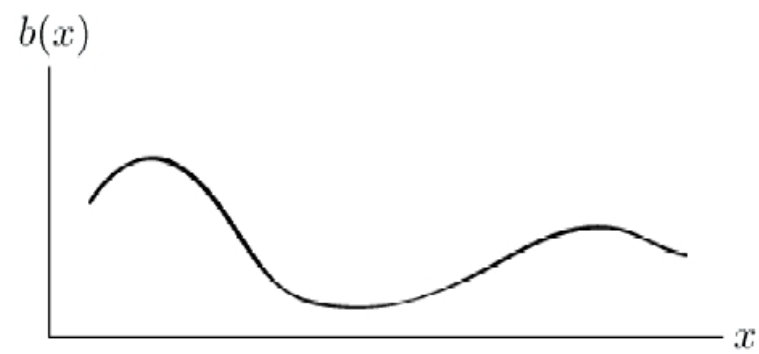
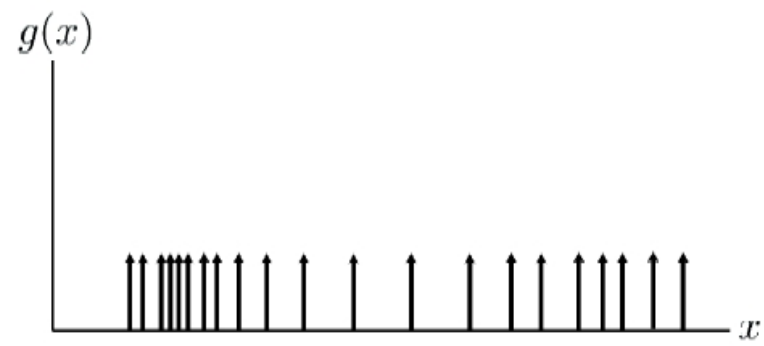
$$\Pr(1 \text{ in } \Delta A) = b(\mathbf{r})\Delta A, \quad (11.73)$$

(c) The probability of more than one count in a small area ΔA is zero:

$$\Pr(1 \text{ in } \Delta A) + \Pr(0 \text{ in } \Delta A) = 1. \quad (11.74)$$

Note that we have allowed $\Pr(1 \text{ in } \Delta A)$ to be a function of position, but $b(\mathbf{r})$ in (11.73) is a fixed function and not yet a random process.

Illustration of fluence and random process



Properties of spatial Poisson processes

$$\text{pr}(\mathbf{r}_n) = \frac{b(\mathbf{r}_n)}{\int_A d^2r b(\mathbf{r})} ; \quad (11.76)$$

$$\text{Pr}(N) = \frac{\overline{N}^N}{N!} \exp(-\overline{N}) ; \quad (11.77)$$

$$\overline{N} = \int_A d^2r b(\mathbf{r}) . \quad (11.78)$$

$$\langle g(\mathbf{r}) \rangle = b(\mathbf{r})$$

$$K_g(\mathbf{r}, \mathbf{r}') = b(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') . \quad (11.94)$$

Key point: All properties determined by $b(\mathbf{r})$, the mean number of counts per unit area or the mean *photon fluence*.

Stochastic Wigner distribution function

For nonstationary spatial Poisson random process, power spectral density is not defined, and not needed since autocorrelation operator is diagonal in the space domain.

If we really have to have a Fourier description, however, we can use the stochastic WDF.

Recall definition from Lecture 13:

$$W_g(\mathbf{r}_0, \boldsymbol{\rho}) = \int_{\infty} d^q \Delta \mathbf{r} \left\langle g(\mathbf{r}_0 + \frac{1}{2} \Delta \mathbf{r}) g^*(\mathbf{r}_0 - \frac{1}{2} \Delta \mathbf{r}) \right\rangle \exp(-2\pi i \boldsymbol{\rho} \cdot \Delta \mathbf{r}) . \quad (8.140)$$

For a spatial Poisson process, we find

$$W_g(\mathbf{r}_0, \boldsymbol{\rho}) = b(\mathbf{r}_0) .$$

In this sense, a Poisson process is “white noise”.

Characteristic functional

PDF of a Poisson process is not defined, and the variance is infinite, but we can describe all statistical properties by a characteristic functional. Recall definition from Lecture 13:

$$\Psi_f(s) = \langle \exp[-2\pi i(s, f)] \rangle, \quad (8.94)$$

where (s, f) is the usual \mathbb{L}_2 scalar product.

For a Poisson random process:

$$\Psi_g(s) = \left\langle \exp \left[-2\pi i \int_A d^2r \, s(\mathbf{r}) \sum_{n=1}^N \delta(\mathbf{r} - \mathbf{r}_n) \right] \right\rangle = \left\langle \exp \left[-2\pi i \sum_{n=1}^N s(\mathbf{r}_n) \right] \right\rangle$$

The expectation is performed in two steps, first over the set of random variables $\{\mathbf{r}_n\}$ for fixed N , then over N . The result is

$$\Psi_g(s) = \exp \left\{ -\overline{N} + \int_A d^2r_n \, b(\mathbf{r}_n) e^{-2\pi i s(\mathbf{r}_n)} \right\}, \quad (11.150)$$

where $b(\mathbf{r})$ is the usual photon fluence (which must be nonrandom for this expression to hold).

Uses of the Poisson characteristic functional

- Compute statistics of filtered Poisson processes
- Develop realistic models of textured objects
- Describe the statistics of digital images with random objects
- Treat difficult (non-Gaussian) speckle problems
- Extend to doubly stochastic case

Doubly stochastic spatial Poisson processes

Suppose fluence is random function.

Important example: Random objects.

$g(\mathbf{r})$ is Poisson for repeated imaging of one object

... but not Poisson when we average over objects.

Autocovariance is modified to:

$$K_g(\mathbf{r}, \mathbf{r}') = \bar{b}(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') + K_b(\mathbf{r}, \mathbf{r}') . \quad (11.116)$$

First term represents the average Poisson random process

Second term is the autocovariance of the fluence $b(\mathbf{r})$.

Summary

- Poisson random variables and processes arise from two different basic assumptions: independence and rarity
- Binomial selection of a Poisson yields a Poisson – and creates one if you didn't have it to start with.
- Poisson random processes are sums of delta functions
- All statistics are contained in the rate or fluence
- Poisson random processes are inherently uncorrelated (unless you insist on using the Fourier domain!)
- Randomness of the rate or fluence spoils the independence and creates correlations