Lecture 3

ADJOINTS AND HERMITIAN OPERATORS

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Motivation:

In quantum mechanics, all physical observables are represented by Hermitian operators, and all measurements yield eigenvalues of the operator

In image science, all linear imaging systems are represented by Hermitian operators, and the set of eigenfunctions and eigenvalues of the operator constitute a complete description of the system.

$\boxed{\text{OUTLINE}}$

- Review of lecture 2

- Scalar products and adjoints

- Examples

- Self-adjoint or Hermitian operators

- Eigenanalysis of Hermitian operators

- Any linear system can be represented by a Hermitian operator – but what happens if it isn't? (Intro to SVD)

$\boxed{\text{OBJECTS}}$

Real-world objects are *functions* $f(\mathbf{r})$, where $\mathbf{r}$ is a q-dimensional vector, e.g. $q = 2$ if $\mathbf{r} = (x, y)$ and $q = 4$ if $\mathbf{r} = (x, y, z, t)$.

Functions can be regarded as vectors in an infinite-dimensional Hilbert space. When we want to take this viewpoint, the function $f(\mathbf{r})$ will be denoted $\mathbf{f}$.

The norm of $\mathbf{f}$ can be defined as

$$||\mathbf{f}\,||^2 \equiv \int_\infty d^q r \, |f(\mathbf{r})|^2 < \infty \, ,$$

and the scalar product of two different functions $f_1(\mathbf{r})$ and $f_2(\mathbf{r})$ (or, equivalently, two different Hilbert-space vectors $\mathbf{f}_1$ and $\mathbf{f}_2$) can be defined by

$$(\mathbf{f}_1, \, \mathbf{f}_2) \equiv \int_\infty d^q r \, f_1^*(\mathbf{r}) \, f_2(\mathbf{r}) \, .$$

The Hilbert space for objects is called *object space* and denoted $\mathbb{U}$.

IMAGES

Digital images are MD vectors. A scalar product between two different digital images (with the same M) is given by

$$(\mathbf{g}_i, \mathbf{g}_j) \equiv \sum_{m=1}^{M} g_{im}^* \, g_{jm} \, .$$

Analog images can be regarded as vectors in an infinite-dimensional Hilbert space, with scalar product defined by

$$(\mathbf{g}_i, \, \mathbf{g}_j) \equiv \int_\infty d^s r_d \, g_i^*(\mathbf{r}_d) \, g_j(\mathbf{r}_d) \, .$$

We denote image space as $\mathbb{V}$, recognizing that it is finite-dimensional for digital images and infinite-dimensional for analog ones.

IMAGING AS MAPPING

Objects are vectors in the infinite-dimensional Hilbert space $\mathbb{U}$

Images are vectors in the Hilbert space $\mathbb{V}$, which is finite-dimensional for digital images and infinite-dimensional for analog ones.

Thus an imaging system *maps* or *transduces* an object to an image. In the absence of noise, the image is related to the object by

$$\mathbf{g} = \boldsymbol{\mathcal{H}}\mathbf{f} \,,$$

where $\boldsymbol{\mathcal{H}}$ is an operator describing the imaging system.

The mathematicians would write

$$\boldsymbol{\mathcal{H}} : \mathbb{U} \to \mathbb{V} \,,$$

meaning that $\boldsymbol{\mathcal{H}}$ operates on a vector in $\mathbb{U}$ (i. e., an object) and produces a vector in $\mathbb{V}$ (an image). $\mathbb{U}$ is the *domain* of $\boldsymbol{\mathcal{H}}$, and $\mathbb{V}$ is its *range*.

CATEGORIES OF OPERATORS

- Continuous-to-continuous (CC) operators

  Lenses, mirrors, etc.
  Computed tomography system, including reconstruction and display

- Discrete-to-discrete (DD) operators

  Matrix model for an imaging system

- Continuous-to-discrete (CD) operators

  CCD camera
  Computed tomography, mapping from object to raw projection data

- Discrete-to-continuous (DC) operators

  Display
  Computer-generated holography

$\boxed{\text{Linear mappings}}$

Linear CC mapping, qD function to sD function:

$$g(\mathbf{r}_d) = \int_{\mathbf{S}_f} d^q r \, h(\mathbf{r}_d, \mathbf{r}) \, f(\mathbf{r}) \, .$$

Linear DD mapping, ND to MD vector:

$$g_m = \sum_{n=1}^{N} H_{mn} \theta_n \, .$$

Linear CD mapping, qD function to MD vector:

$$g_m = \int_{\mathbf{S}_f} d^q r \, h_m(\mathbf{r}) \, f(\mathbf{r})$$

S_f is object support (may be infinite)

PRF for linear CD systems

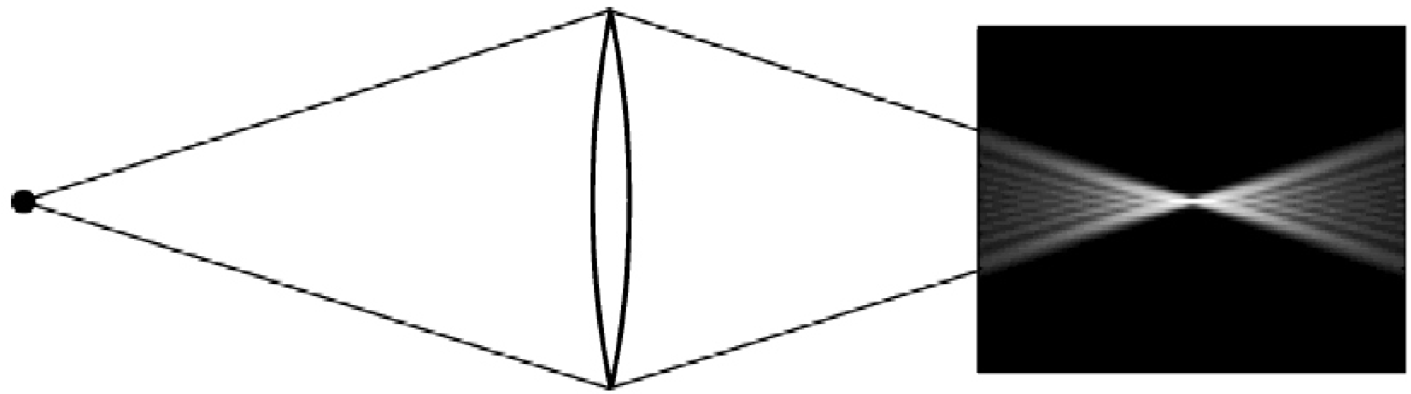
$$g_m = \int_{\mathbf{S}_f} d^q r \, h_m(\mathbf{r}) \, f(\mathbf{r})$$

The kernel $h_m(\mathbf{r})$ is the point response function (PRF) of the system. For a delta-function input, the output is

$$g_m = \int_{\mathbf{S}_f} d^q r \, h_m(\mathbf{r}) \, \delta(\mathbf{r} - \mathbf{r}_0) = h_m(\mathbf{r}_0) \, .$$

Another name for $h_m(\mathbf{r})$ is the *sensitivity function* since it specifies how sensitive the m^{th} discrete detector element is to a point source at $\mathbf{r} = \mathbf{r}_0$.

3D point response function of a lens



This picture can represent:

CC PRF of lens as 3D → 2D mapping
Sensitivity function
CD PRF (if point on the left is a discrete detector element)

Adjoints of linear operators

General linear operator

$$\boldsymbol{\mathcal{H}} : \mathbb{U} \rightarrow \mathbb{V}$$

Adjoint operator

$$\boldsymbol{\mathcal{H}}^\dagger : \mathbb{V} \rightarrow \mathbb{U}$$

Inverse operator (if it exists)

$$\boldsymbol{\mathcal{H}}^{-1} : \mathbb{V} \rightarrow \mathbb{U}$$

BUT

- Adjoint is not the same as inverse in general

- Inverse does not exist for most operators (adjoint always exists)

Definition of adjoint

Ref: Barrett and Myers, Sec. 1.3.7

Basic *definition*:

$$(\mathbf{g}_2, \boldsymbol{\mathcal{H}}\mathbf{f}_1)_{\mathbb{V}} = (\boldsymbol{\mathcal{H}}^{\dagger}\mathbf{g}_2, \mathbf{f}_1)_{\mathbb{U}} \tag{1.39}$$

Note that scalar product on the left is in image space, the one on the right is in object space

It is legal *by definition* to move an operator to the other side in a scalar product and replace it by its adjoint.

Properties of adjoints

The following properties of adjoints follow from the definition and properties of scalar products:

(*a*) $(c\boldsymbol{\mathcal{H}})^\dagger = c^*\boldsymbol{\mathcal{H}}^\dagger$;

(*b*) $(\boldsymbol{\mathcal{H}}_1 + \boldsymbol{\mathcal{H}}_2)^\dagger = \boldsymbol{\mathcal{H}}_1^\dagger + \boldsymbol{\mathcal{H}}_2^\dagger$;

(*c*) $(\boldsymbol{\mathcal{H}}_1\boldsymbol{\mathcal{H}}_2)^\dagger = \boldsymbol{\mathcal{H}}_2^\dagger\boldsymbol{\mathcal{H}}_1^\dagger$;

(*d*) $(\boldsymbol{\mathcal{H}}^\dagger)^\dagger = \boldsymbol{\mathcal{H}}$,

where c is a scalar and $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$ are two different operators mapping $\mathbb{U}$ to $\mathbb{V}$.

Terminology

If $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}^{\dagger}$, which is possible only if $\mathbb{U} = \mathbb{V}$, the operator is said to be self-adjoint or *Hermitian*.

If $\boldsymbol{\mathcal{H}}^{\dagger} = \boldsymbol{\mathcal{H}}^{-1}$, again possible only if $\mathbb{U} = \mathbb{V}$, the operator is said to be *unitary*.

Example: Adjoint of a matrix operator

If $\boldsymbol{\mathcal{H}}$ maps $\mathbb{E}^N$ to $\mathbb{E}^M$, it is represented by an $M \times N$ matrix $\mathbf{H}$. Hence $\boldsymbol{\mathcal{H}}^\dagger$ is represented by an $N \times M$ matrix $\mathbf{H}^\dagger$.

$$(\mathrm{g}_2, \boldsymbol{\mathcal{H}}\mathrm{f}_1)_{\mathbb{V}} = (\boldsymbol{\mathcal{H}}^\dagger \mathrm{g}_2, \mathrm{f}_1)_{\mathbb{U}} \tag{1.39}$$

From left-hand side of (1.39), we obtain

$$(\mathrm{g}_2, \boldsymbol{\mathcal{H}}\mathrm{f}_1)_{\mathbb{V}} = \sum_{m=1}^{M} g_{2m}^* \sum_{n=1}^{N} H_{mn} f_{1n} = \sum_{n=1}^{N} \sum_{m=1}^{M} g_{2m}^* H_{mn} f_{1n} \,.$$

Similarly, the right-hand side of (1.39) yields

$$(\boldsymbol{\mathcal{H}}^\dagger \mathrm{g}_2, \mathrm{f}_1)_{\mathbb{U}} = \sum_{n=1}^{N} \left[\sum_{m=1}^{M} \left[\mathbf{H}^\dagger\right]_{nm} g_{2m} \right]^* f_{1n} = \sum_{n=1}^{N} \sum_{m=1}^{M} \left[\mathbf{H}^\dagger\right]_{nm}^* g_{2m}^* f_{1n} \,.$$

Comparison of the final forms shows that

$$[\mathbf{H}^\dagger]_{nm} = H_{mn}^*$$

$\boxed{\text{Adjoint of a matrix operator – summary}}$

We just showed that

$$[\mathbf{H}^\dagger]_{nm} = H^*_{mn}\,,$$

so the adjoint of $\mathbf{H}$ is obtained by transposing it (interchanging rows and columns) and taking the complex conjugate of each element.

$$\text{Adjoint} = \text{complex conjugate of transpose}$$

For real matrices:

$$\text{Adjoint} = \text{transpose}$$

Superscript t denotes transpose, superscript $\dagger$ denotes adjoint

Adjoints of CC and CD operators

CC operator

$$[\boldsymbol{\mathcal{H}}\mathbf{f}](\mathbf{r}_d) = g(\mathbf{r}_d) = \int_\infty d^q r \; h(\mathbf{r}_d, \mathbf{r})\, f(\mathbf{r})$$

Adjoint

$$\left[\boldsymbol{\mathcal{H}}^\dagger \mathbf{g}\right](\mathbf{r}) \int_\infty d^s r_d \; h^{(\dagger)}(\mathbf{r}, \mathbf{r}_d)\, g(\mathbf{r}_d) = \int_\infty d^s r_d \, h^*(\mathbf{r}_d, \mathbf{r})\, g(\mathbf{r}_d)$$

So kernel of the adjoint operator, $h^{(\dagger)}(\mathbf{r}, \mathbf{r}_d) = h^*(\mathbf{r}_d, \mathbf{r})$

CD operator

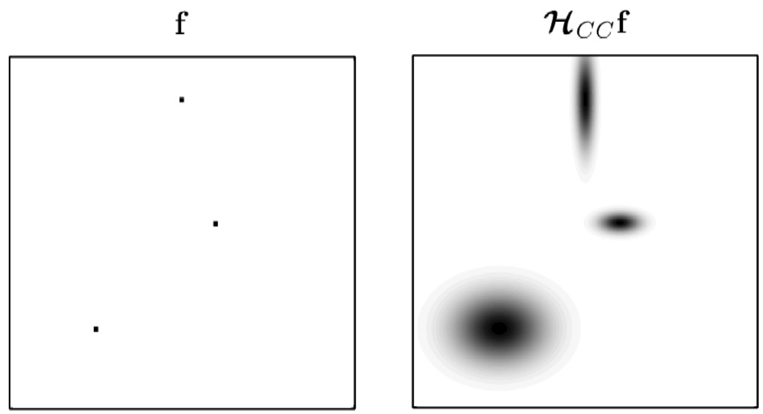
$$[\boldsymbol{\mathcal{H}}\mathbf{f}]_m = g_m = \int_\infty d^q r \; h_m(\mathbf{r})\, f(\mathbf{r})$$

Adjoint

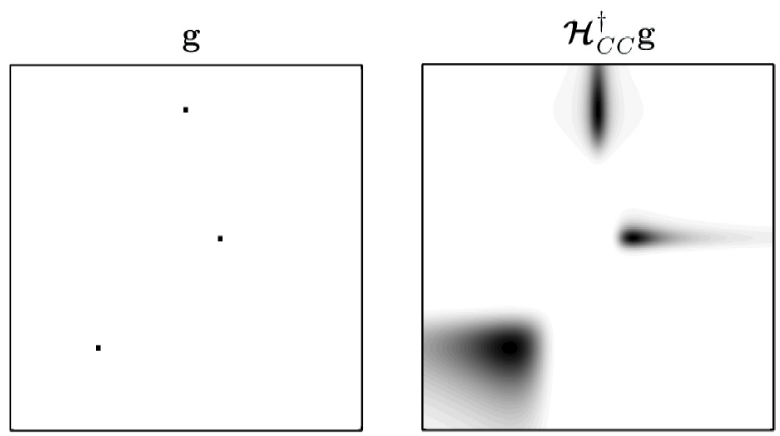
$$\left[\boldsymbol{\mathcal{H}}^\dagger \mathbf{g}\right](\mathbf{r}) = \sum_{m=1}^{M} h_m^*(\mathbf{r})\, g_m$$

Continuous-to-continuous (CC) operator

Forward operator

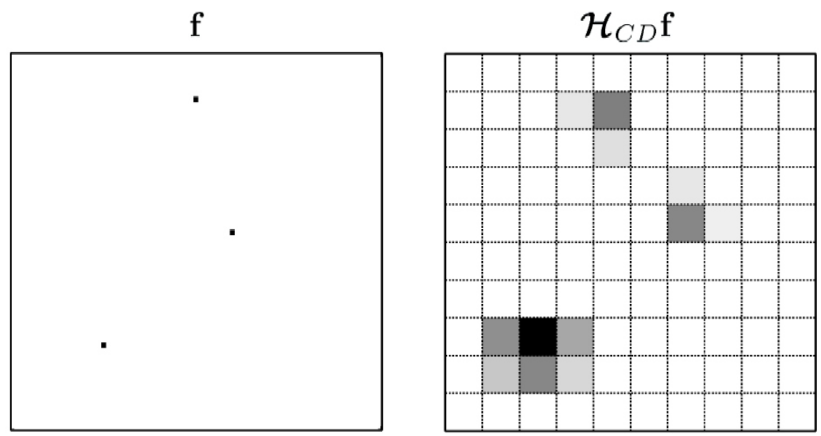


Adjoint operator

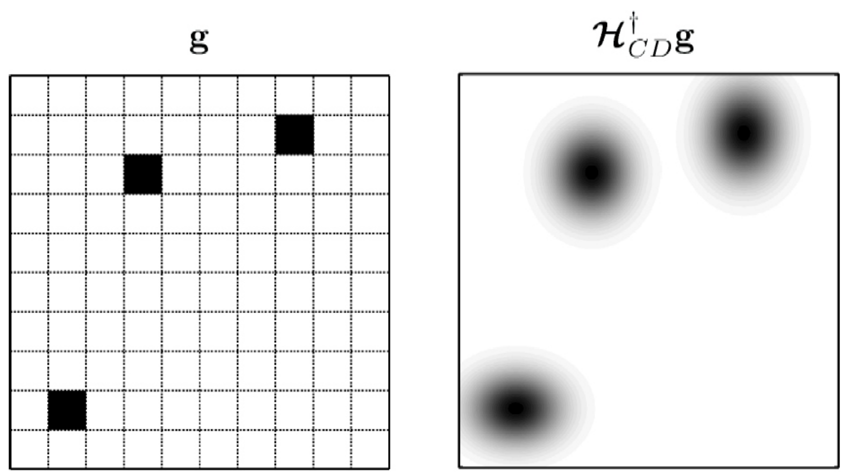


Continuous-to-discrete (CD) operator

Forward operator



Adjoint operator



Conditions for an operator to be Hermitian

A CD operator is *never* Hermitian ($\mathbb{U} \neq \mathbb{V}$)

A DD operator with a non-square matrix ($M \neq N$) is never Hermitian

A square matrix is Hermitian if

$$H_{mn} = H_{nm}^*$$

A CC operator mapping a qD function to a sD function cannot be Hermitian if $q \neq s$.

If $q = s$, a CC operator with kernel $h(\mathbf{r}_d, \mathbf{r})$ is Hermitian if

$$h(\mathbf{r}_d, \mathbf{r}) = h^*(\mathbf{r}, \mathbf{r}_d)$$

Thus LSIV systems are Hermitian if the PSF is real and symmetric (e.g., defocus, spherical aberration) but not if the PSF is asymmetric (coma)

Construction of Hermitian operators

Most linear operators of interest in imaging are not Hermitian, either because range $\neq$ domain or because kernel is not symmetric.

What to do?

If nature doesn't give you a Hermitian operator, make one!

If $\boldsymbol{\mathcal{H}}$ is an arbitrary linear operator, then $\boldsymbol{\mathcal{H}}^\dagger\boldsymbol{\mathcal{H}}$ and $\boldsymbol{\mathcal{H}}\boldsymbol{\mathcal{H}}^\dagger$ are Hermitian.

Proof uses $(\boldsymbol{\mathcal{H}}_1\boldsymbol{\mathcal{H}}_2)^\dagger = \boldsymbol{\mathcal{H}}_2^\dagger\boldsymbol{\mathcal{H}}_1^\dagger$ and $(\boldsymbol{\mathcal{H}}^\dagger)^\dagger = \boldsymbol{\mathcal{H}}$

Terminology from the tomography literature:

$\boldsymbol{\mathcal{H}}$ is the projection operator

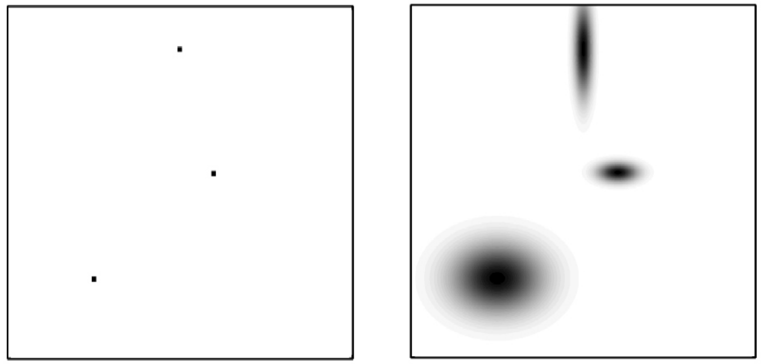
$\boldsymbol{\mathcal{H}}^\dagger$ is the backprojection operator

$\boldsymbol{\mathcal{H}}^\dagger\boldsymbol{\mathcal{H}}$ is the projection/backprojection operator

Continuous-to-continuous (CC) operator

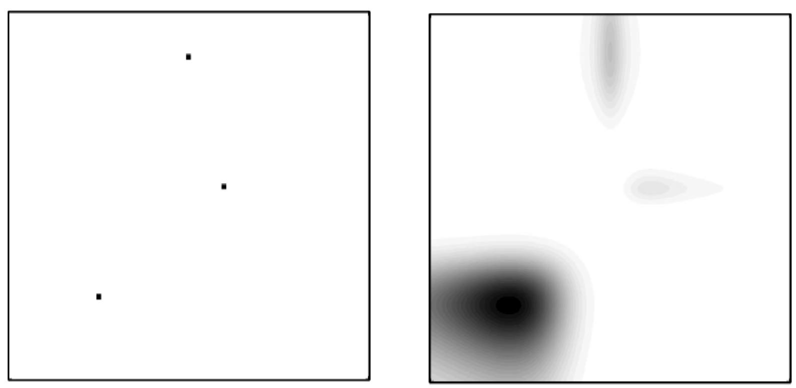
Projection
(forward operator)

$\mathbf{f}$ $\quad$ $\mathcal{H}_{CC}\mathbf{f}$



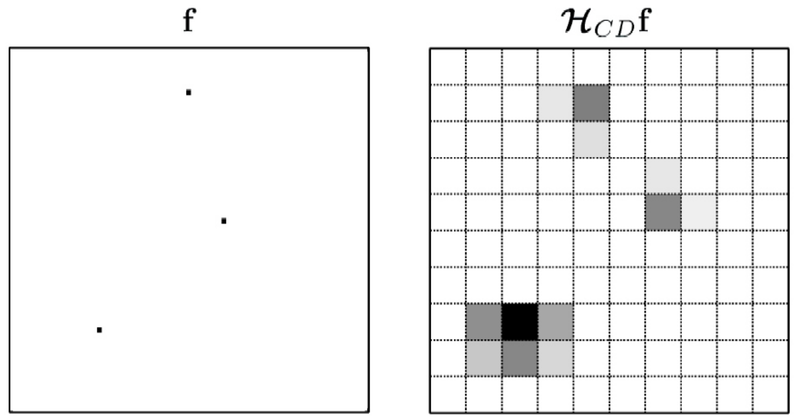
Projection/backprojection

$\mathbf{f}$ $\quad$ $\mathcal{H}^{\dagger}_{CC}\mathcal{H}_{CC}\mathbf{f}$

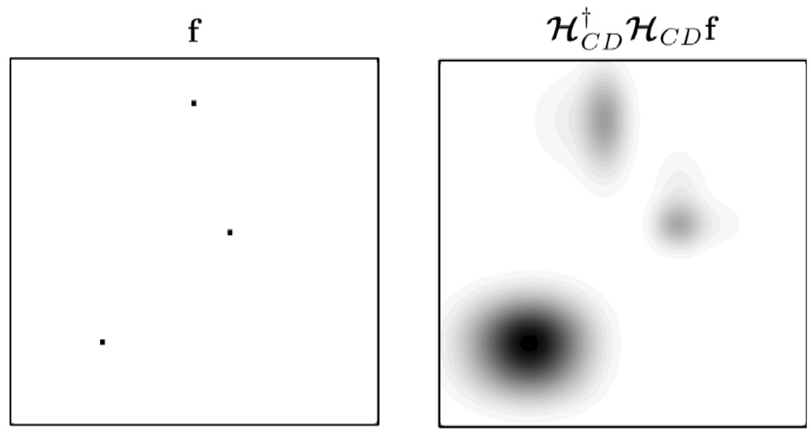


Continuous-to-discrete (CD) operator

Projection
(forward operator)



Projection/backprojection



Eigenalysis of Hermitian operators

Q. Why do we care so much about Hermitian operators?

A. Because their eigenvectors and eigenvalues are magical!
(Ref: Barrett and Myers, Sec. 1.4)

Eigenvalue equation for a general linear operator:

$$\boldsymbol{\mathcal{A}}\boldsymbol{\psi} = \lambda \boldsymbol{\psi}\,, \tag{1.65}$$

There may be no solution, a finite set of solutions, a countably infinite set of solution or an uncountable set of solutions. If the solutions are countable, we can write

$$\boldsymbol{\mathcal{A}}\boldsymbol{\psi}_n = \lambda_n \boldsymbol{\psi}_n\,. \tag{1.67}$$

If $\boldsymbol{\mathcal{A}}$ is a *compact* Hermitian operator, then:

The solutions form a countable set
The number of independent solutions is the dimension of the space
The eigenvalues are always real

Other nifty properties of eigenvalues and eigenvectors of Hermitian operators

- Eigenvectors are orthogonal: $(\psi_n, \psi_m) = \delta_{nm}$

- Eigenvectors form a basis for the domain of the operator.

- Arbitrary vector in domain can be written as

$$\mathbf{f} = \sum_{n=1}^{N} \alpha_n \boldsymbol{\psi}_n \,, \qquad \alpha_n = (\boldsymbol{\psi}_n, \mathbf{f}) \,,$$

where N is the dimension of the space (may be infinity)

In the next lecture, we will apply these properties to imaging systems and construct the singular value decomposition of a linear (but not Hermitian) system operator.