SUPPERRESOLUTION

Harrison H. Barrett
University of Arizona

What is it?

Why is it possible?

Is it practical?

What is superresolution?

Many optical systems have zero response to spatial frequencies beyond a cutoff value; these frequencies correspond to null functions of the system.

Suppose there is no noise and no image sampling, so that the frequency components below the cutoff are known perfectly.

Can we then recover the object exactly at all spatial frequencies?

If not, can we at least determine the object Fourier transform for *some* frequencies beyond the cutoff?

If the answer to either question is yes, we can perform *superresolution* or *bandwidth extrapolation*.

Prior information

If we know nothing at all about the object ahead of time (*a priori*), no bandwidth extrapolation at all is possible.

Useful forms of prior knowledge include:

- Finite spatial support
- Some measure of smoothness
- Positivity
- Sparseness (object is zero over most of its support)
- A parametric description

  Example: Set of N points of unknown strength and location

Digression: Analytic functions

Reference: Barrett and Myers, Appendix B, Complex Variables

Consider a scalar-valued function of a complex variable z:

$$z = x + iy\,, \qquad f(z) = u(x,y) + iv(x,y)\,, \qquad x, y, u, v \text{ all real}$$

The derivative of this function is defined by

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}\,. \tag{B.28}$$

The function $f(z)$ is differentiable in any region where this limit is finite *and* independent of the direction of Δz in the complex plane.

$\boxed{\text{Terminology}}$

A function $f(z)$ is said to be *analytic* at z_0 if its derivative exists at every point in some neighborhood of z_0 (including the point z_0 itself).

Synonyms: *analytic*, *regular* and *holomorphic*

If $f(z)$ is analytic at every point in the neighborhood of z_0 but not at z_0 itself, z_0 is called a *singular point* or *singularity* of $f(z)$.

A function that is analytic everywhere (except possibly at ∞) is called *entire*. For example, a polynomial is entire.

$\boxed{\text{Conditions for analyticity}}$

The following conditions are all equivalent. A function $f(z)$ is analytic in some region if and only if:

- Its first derivative exists and is unique (independent of direction)
- *All* of its derivatives exist and are unique.
- The Cauchy-Riemann equations are satisfied:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y}, \qquad \frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x}. \tag{B.31}$$

If CR conditions hold, then the real and imaginary parts, $u(x,y)$ and $v(x,y)$, are *harmonic functions*; *i.e.*, they satisfy the 2D Laplace equation:

$$\nabla^2 u(x,y) = \nabla^2 v(x,y) = 0\,. \tag{B.32}$$

$\boxed{\text{More on analytic functions}}$

- Analytic functions cannot have maxima in their region of analyticity!

- None of the derivatives of an analytic function can have maxima.

- A function that is analytic and bounded for all z must be a constant!

- Zeros of analytic functions can occur only at isolated points. If $f(z)$ is analytic and zero over a finite line or area, it must be zero everywhere!

- The integral of an analytic function over any closed path must be zero; if $f(z)$ is analytic inside and on contour C, then

$$\oint_C dz \; f(z) = 0 \,. \tag{B.38}$$

Still more on analytic functions

• Analytic functions and their derivatives are determined by boundary values:

$$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0} \, . \qquad \text{(B.41)}$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C dz \, \frac{f(z)}{(z - z_0)^{n+1}} \, , \qquad \text{(B.42)}$$

• An analytic function can be expanded in a *Taylor* series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \, , \qquad \text{(B.43)}$$

$$a_n = \frac{1}{n!} f^{(n)}(z_0) = \frac{1}{2\pi i} \oint_C dz' \, \frac{f(z')}{(z' - z_0)^{n+1}} \, . \qquad \text{(B.44)}$$

• If an analytic function is known in any region or on any line, it is known everywhere in its region of analyticity.

$\boxed{\text{Why do we care about analytic functions in imaging?}}$

Analyticity is a very strong condition, an extreme form of smoothness. *None* of the objects of interest in imaging are analytic functions, ...

.... but their Fourier transforms are!

Reference: Barrett and Myers, Chap. 3, Sec. 3.3.9

Complex spatial frequencies

The Fourier transform of a function $f(x)$ defined on the real line is usually stated as

$$F(\xi) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-2\pi i \xi x}\,.$$

The spatial frequency ξ is usually a real variable, but it is also of interest to consider complex frequencies:

$$\xi = \xi_r + i\xi_i\,, \tag{3.200}$$

where subscripts r and i denote real and imaginary parts, respectively. The Fourier transform now reads

$$F(\xi) = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-2\pi i \xi_r x}\, e^{2\pi \xi_i x}\,. \tag{3.201}$$

Question: Is $F(\xi)$ analytic?

$\boxed{\text{Paley-Wiener theorem}}$

Q. Is $F(\xi)$ analytic?
A. Yes, if $f(x)$ is bounded and has finite support.

Suppose the support of $f(x)$ is $(-\frac{1}{2}L, \frac{1}{2}L)$, so that

$$F(\xi) = F_r(\xi_r, \xi_i) + iF_i(\xi_r, \xi_i)$$

$$= \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx\, f(x)\, e^{2\pi\xi_i x} \cos(2\pi\xi_r x) - i \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx\, f(x)\, e^{2\pi\xi_i x} \sin(2\pi\xi_r x)\,.$$

By differentiating under the integral sign (legal for finite support), we can show that the Cauchy-Riemann conditions are satisfied:

$$\frac{\partial F_r}{\partial \xi_r} = \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx\, f(x)\, (-2\pi x) \sin(2\pi\xi_r x)\, e^{2\pi\xi_i x} = \frac{\partial F_i}{\partial \xi_i}\,, \qquad (3.205)$$

and similarly for the second condition.

Thus $F(\xi)$ is *entire* (analytic for all finite ξ) if $f(x)$ is bounded and has compact support (Paley and Wiener, 1934).

$\boxed{\text{Back to imaging!}}$

Paley and Wiener tell us that bandwidth extrapolation is possible, but how do we do it?

One answer: Use prolate spheroidal wavefunctions.

Another digression: Prolate spheroidal wavefunctions

Reference: Barrett and Myers, Chap. 4, Sec. 4.1.5

Prolate spheroidal wavefunctions are eigenfunctions of *two* Hermitian operators, with different domains:

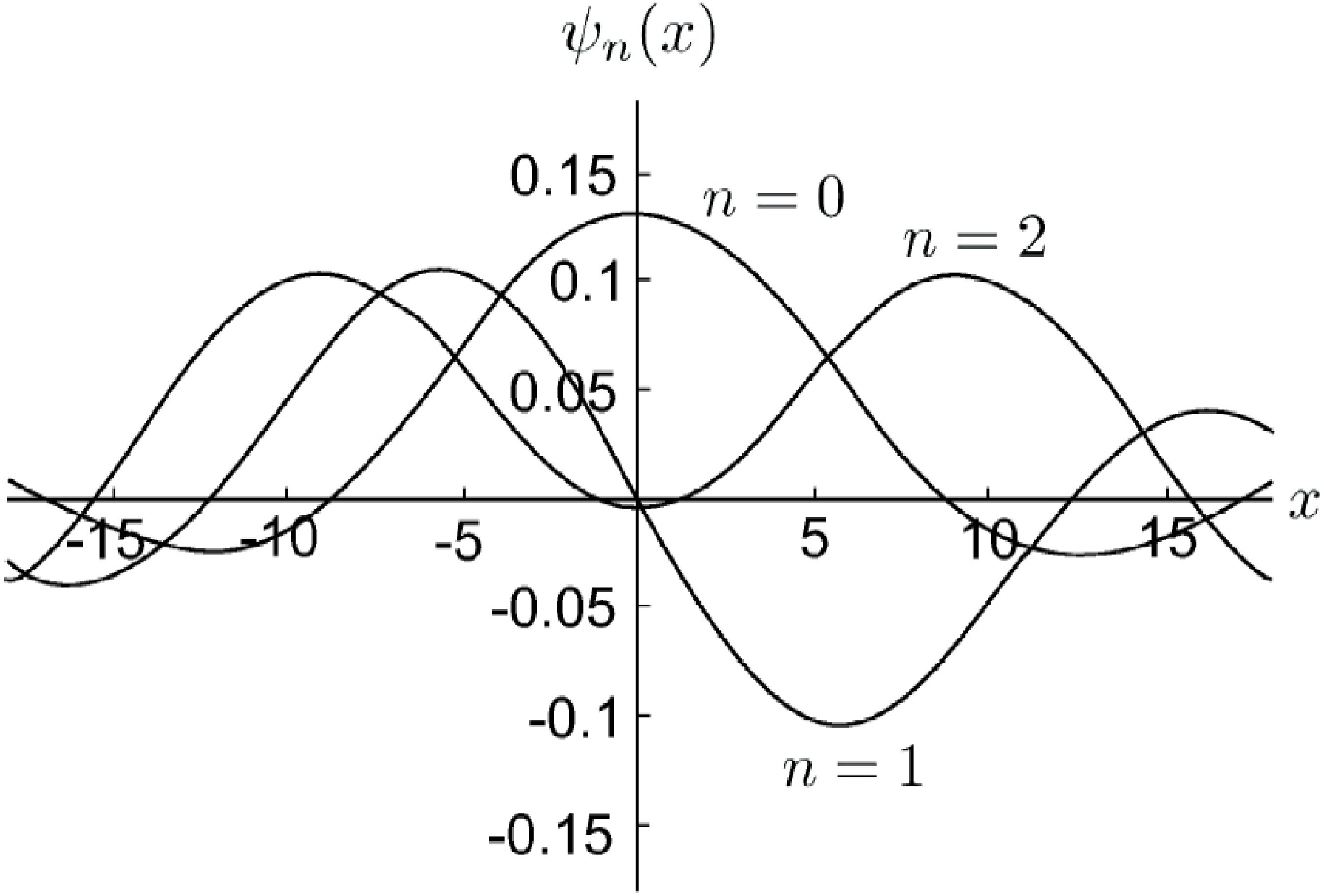
- The Helmholtz equation after separation in prolate spheroidal coordinates
  Domain: $-\infty < x < \infty$

- Truncation followed by low-pass filtering:
  Domain: $-\frac{1}{2}L < x < \frac{1}{2}L$

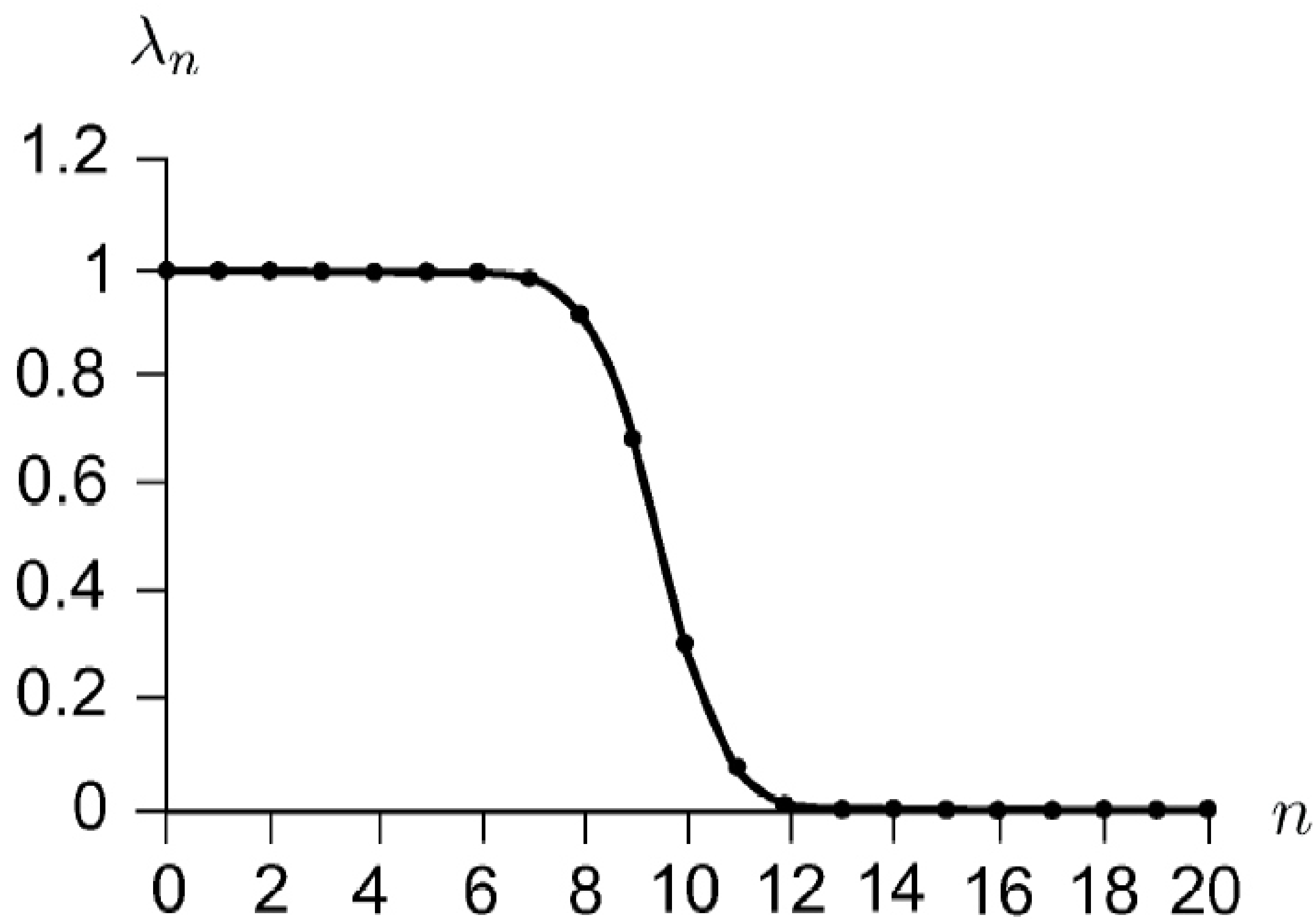
$$B \int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx' \, \psi(x') \, \mathrm{sinc}[B(x - x')] = \lambda_n \psi_n(x) \, . \tag{4.64}$$

$$\boldsymbol{\mathcal{B}}_B \boldsymbol{\mathcal{S}}_L \psi_n(x) = \lambda_n \psi_n(x) \, , \tag{4.65}$$

The first three prolate spheroidal wavefunctions



Eigenvalues associated with space-limiting followed by band-limiting
(Space-bandwidth product = 10)



Orthogonality and completeness of prolates

- Orthonormal on the infinite interval:

$$\int_{-\infty}^{\infty} dx \; \psi_n(x)\,\psi_m(x) = \delta_{nm}\,, \tag{4.66}$$

- Orthogonal but not normalized on the finite interval:

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} dx \; \psi_n(x)\,\psi_m(x) = \lambda_n\,\delta_{nm}\,. \tag{4.67}$$

- Complete on the finite interval

- Bandlimited, therefore *not* complete on the infinite interval

- Complete in the space of bandlimited, square-integrable functions on the real line (Paley-Wiener space).

$\boxed{\textit{Now}\text{ we're back to imaging!}}$

1D object is known to be space-limited: $f(x) = f(x)\ \mathrm{rect}(x/L) = \boldsymbol{\mathcal{S}}_L f(x)$.

Imaging system is ideal, noise-free, LSIV – but bandlimited:

$$g(x) = [h * f](x)\,, \qquad G(\xi) = H(\xi)\,F(\xi)\,, \qquad H(\xi) = H(\xi)\ \mathrm{rect}(\xi/B)\,.$$

Get modified data by Fourier-domain inverse filtering:

$$\tilde{G}(\xi) \equiv \frac{G(\xi)}{H(\xi)} = F(\xi)\ \mathrm{rect}(\xi/B)\,, \qquad \tilde{g}(x) = \boldsymbol{\mathcal{B}}_B f(x) = \boldsymbol{\mathcal{B}}_B \boldsymbol{\mathcal{S}}_L f(x)\,.$$

Now expand $f(x)$ in prolates and apply the operator $\boldsymbol{\mathcal{B}}_B \boldsymbol{\mathcal{S}}_L$:

$$f(x) = \sum_{n=0}^{\infty} \alpha_n\,\psi_n(x)\,, \qquad \tilde{g}(x) = \sum_{n=0}^{\infty} \beta_n\,\psi_n(x) = \sum_{n=0}^{\infty} \alpha_n\,\lambda_n\,\psi_n(x)\,.$$

Coefficients $\{\beta_n\}$ can be found by use of orthonormality, (4.66). Final answer is obtained by prolate-domain inverse filtering:

$$f(x) = \sum_{n=0}^{\infty} \frac{\beta_n}{\lambda_n}\,\psi_n(x)\,,$$

Problems

- You never have continuous data.

- If you did, how would you do the Fourier-domain inverse filtering?

- $\lambda_n \to 0$ very rapidly for $n > LB$.

- You never have noise-free data, so dividing by small eigenvalues is a problem in the prolate-domain inverse filtering.

- Even if you have noise-free data, and even if data sampling is ignored, computer precision will catch up with you pretty soon.

$\boxed{\text{The importance of being positive}}$

(Slide title stolen from Roy Frieden)

Bandlimited imaging systems have null functions:

$$f(x) = f_{meas}(x) + f_{null}(x)\,,$$

$$F_{meas}(\xi) = 0 \text{ for } |\xi| > \tfrac{1}{2}B\,, \qquad F_{null}(\xi) = 0 \text{ for } |\xi| < \tfrac{1}{2}B\,.$$

We often know *a priori* that $f(x) \geq 0, \forall x$, but $f_{meas}(x)$ and $f_{null}(x)$ will almost always take on negative values.

Knowledge that $f(x) \geq 0$ greatly reduces our freedom for putting stuff in the null space. (Details in Chap. 15 of B&M).

Moreover, use of positivity leads to a practical algorithm for at least partial bandwidth extrapolation.

$\boxed{\text{Convex sets}}$

A satisfactory solution to the bandwidth-extrapolation problem should:

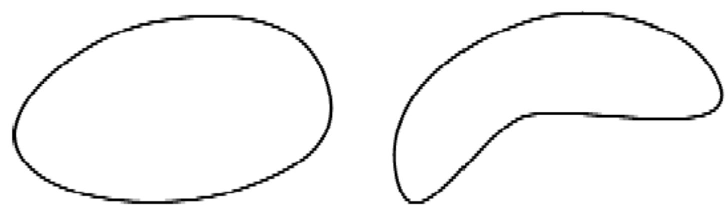
- Agree with the data, i.e., give the correct $f_{meas}(x)$
- Be space-limited
- Be non-negative

Each of these constraints defines a *convex set*.

Another digression: Definition of convex sets

Reference: B&M, Chap. 15

If $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ are members of the set, so is $\alpha\boldsymbol{\theta} + (1-\alpha)\boldsymbol{\theta}'$ for $0 \le \alpha \le 1$. Geometric interpretation: all points along the line joining $\boldsymbol{\theta}$ and $\boldsymbol{\theta}'$ are members of the set.



Convex set Nonconvex set

By this definition, we have three convex sets in bandwidth-extrapolation:

- Set of all functions with same low-frequency components
- Set of all functions such that $f(x) = f(x)\ \mathrm{rect}(x/L)$
- Set of all non-negative functions

Projections onto convex sets

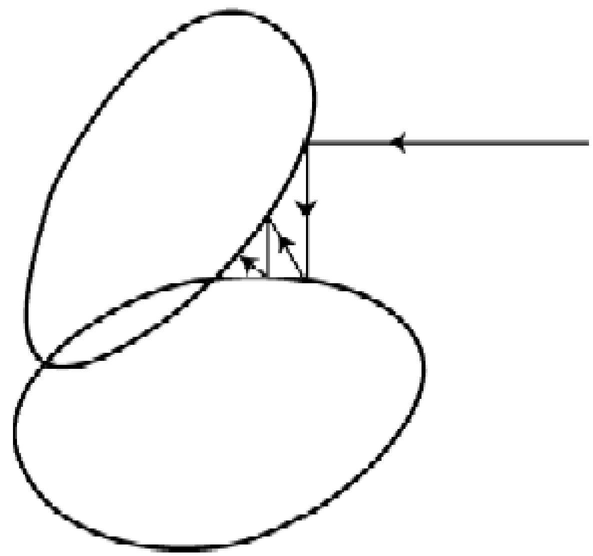
To “project” onto the set of non-negative functions, set any neagative values to zero

To project onto the set of functions with a specified support, set the function to zero outside

To project onto the set of functions with same LF components, replace existing LF components with the prescribed values

Projections onto convex sets – cont.

The POCS algorithm is just to project onto each set sequentially:



Lots of theorems guarantee that the final result will be an image consistent with all convex constraints. As applied to bandwidth extrapolation, the POCS algorithm is called Gerchberg-Papoulis.

Summary

• Elegant theorems about analyticity of Fourier transforms indicate that bandwidth extrapolation is possible,....

... but don't help you achieve it!

• POCS and other iterative algorithms can achieve some extrapolation, ...

... but not much!

• How much depends on noise level, size of object support and sparsity of object.