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I

THE STATISTICS OF SPECKLE PATTERNS

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CONTENTS

§ 1. Introduction

Light with a fair degree of spatial and temporal coherence incident on an optically rough surface produces a reflected or transmitted beam that has a random spatial variation of intensity. This intensity distribution is called a speckle pattern. Fig. 1 shows the speckle pattern obtained at a distance of 1 m from a 1 mm diameter area of ground glass illuminated by a He–Ne laser. In this case the speckle pattern is of high contrast, and has a characteristic scale, or speckle "size", approximately equal to the diameter of the Airy disc that would be produced in the absence of the ground glass (i.e. approximately 1 mm).

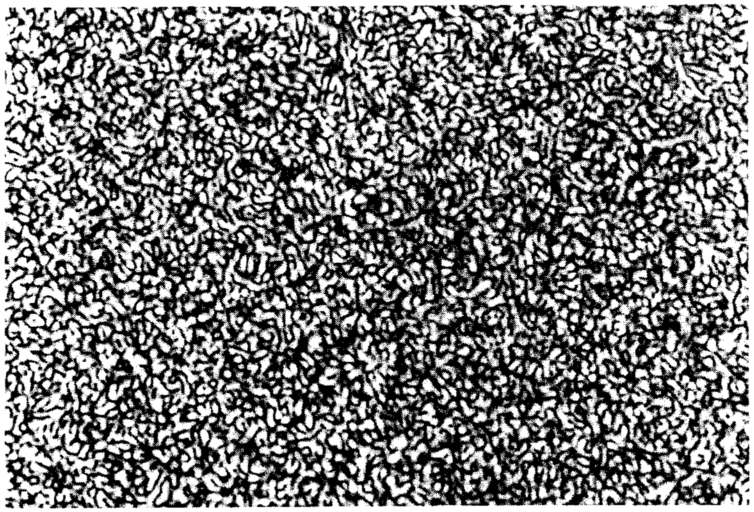

Fig. 1. Speckle pattern produced in the Fraunhofer plane of an optically rough diffuser illuminated by a He–Ne laser.

The statistical properties of a speckle pattern depend, *in general*, on both the coherence of the incident light and the statistics of the scattering surface or medium. In the laboratory, speckle patterns are usually produced by

highly coherent light incident on relatively large areas of optically very rough surfaces and this case is in fact an exception to the general rule; the statistics of such speckle patterns do *not* depend on the detailed surface properties and we shall refer to these as *normal* speckle patterns. The statistics of such spatial patterns are very closely related to those of the temporal fluctuations of thermal (Gaussian) light sources (JAKEMAN [1974]). The general dependence of speckle statistics on the coherence of the incident light and the nature of the scatterer has led to several applications in the measurement of coherence and scattering parameters.

One of the first recorded observations of a speckle pattern was by EXNER [1877, 1880] who sketched the form of the pattern produced by candlelight incident on a glass plate on which he had breathed. The non-monochromaticity of this light source caused the pattern to have a radially fibrous structure and this feature was extensively discussed in the early literature (VON LAUE [1914, 1916, 1917], DE HAAS [1918a, b], BUCHWALD [1919], RAMAN [1919], RAMACHANDRAN [1954]). The mathematical basis for much of the analysis of the statistics of speckle patterns was established by Lord RAYLEIGH [1880, 1918, 1919]. The detailed first and second order statistics of normal speckle patterns formed in the Fraunhofer plane were fully evaluated by VON LAUE [1914, 1916]; in particular he calculated general expressions for the second order probability density function of the intensity and the autocorrelation function of the intensity. Early work on speckle patterns is reviewed by HARIHARAN [1972].

Following the invention of the laser the phenomenon of speckle was re-discovered and a large number of short papers describing various simple properties were published. These are included in a bibliography on speckle compiled by SINGH [1972].

In this article we shall concentrate our attention on the first and second order statistics of speckle patterns. § 2 is concerned with normal speckle patterns formed in perfectly coherent light. In this section, patterns formed in the image plane and in the Fraunhofer plane of a diffuser are considered separately, although of course a general theory could be used to cover both cases; with the appropriate assumptions in each case, the statistics are in fact identical. The statistics in the near-field are not evaluated as they are essentially the same as those in the far-field except for regions very close to the diffuser (ELIASSON and MOTTIER [1971]). The effect of partially coherent illumination on speckle statistics is discussed in § 3, and surface dependent aspects are considered in § 4.

Random intensity patterns produced by volume scatterers such as the atmosphere are not explicitly included in this article. The question of propagation through scattering media is very much more complex than the relatively simple cases described here and has only partially been solved (CHERNOV [1960], TATARSKI [1961], STROHBEHN [1971]). Finally, many of the details of the scattering process and the nature of the scattering surface are excluded from the discussion below and are described by BECKMANN and SPIZZICHINO [1963] and BECKMANN [1967].

§ 2. Normal Speckle Patterns

2.1. FIRST ORDER STATISTICS

We shall consider first a speckle pattern formed in coherent light in the Fraunhofer plane, as shown in Fig. 2. The effective complex amplitude of

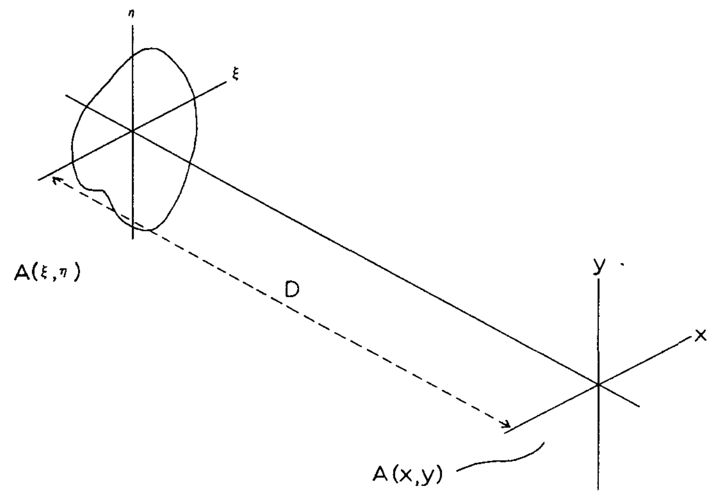

Fig. 2. Formation of a speckle pattern in the Fraunhofer plane.

the scattered light in the scattering plane may be written as

$$A(\xi, \eta) = \sum_{j=1}^{N} a_j \exp(\mathrm{i}\beta_j)\delta(\xi-\xi_j)\delta(\eta-\eta_j), \tag{1}$$

where N is the number of independent scatterers,

a_j is the modulus of the scattered wave due to the jth scatterer,

β_j is the phase of the scattered wave,

and $\delta(\xi)\delta(\eta)$ is the two-dimensional Dirac delta function.

The complex amplitude $A(\xi, \eta)$ is a random process with the following assumptions made (GOODMAN [1963, 1975c]):

i) the scatterers are randomly distributed over the area of the diffuser with uniform probability,
ii) the a_j are statistically independent random variables,
iii) the β_j are statistically independent random variables, are uniformly distributed in the interval $-\pi$ to π (*rough surface approximation*) and are independent of the a_j,
iv) the polarisation of the incident wave is unaltered.

It should be noted that these assumptions apply to the effect of the scattering medium on the incident field thus circumventing a detailed consideration of the interaction of the field and scatterer. In the so-called Beckmann model (BECKMANN and SPIZZICHINO [1963]) the statistical properties of the scattering surface are specified and the field immediately after the scatterer is deduced making appropriate assumptions about the interaction. This more general approach however can only be pursued in detail for scatterers with a Gaussian distribution of surface heights and does not indicate the conditions under which we might expect surface-independent (normal) speckle patterns. It can be shown that for a Gaussian distribution of surface heights with an autocorrelation function whose scale width is very much less than the overall dimensions of the scatterer the Beckmann model gives the same result as the Goodman model for the statistics of the speckle intensity fluctuation; in addition it also predicts the shape of the envelope of the average intensity in the Fraunhofer plane.

Making the usual far field assumptions, the complex amplitude in the observing plane $A(x, y)$ may be written as,

$$A(x, y) = \int_{-\infty}^{\infty}\int A(\xi, \eta) \exp\left(-\frac{2\pi \mathrm{i}}{\lambda D}(x\xi + y\eta)\right) \mathrm{d}\xi\, \mathrm{d}\eta,$$

where the unimportant phase factor has been ignored. Substituting for $A(\xi, \eta)$ and evaluating the Fourier transform we obtain,

$$A(x, y) = \sum_{j=1}^{N} a_j \exp(\mathrm{i}\beta_j) \exp\left(-\frac{2\pi \mathrm{i}}{\lambda D}(x\xi_j + y\eta_j)\right). \tag{2}$$

It is clear from this expression that the complex amplitude in the observing plane is given by the sum of a large number of random phase and amplitude vectors. As a result of the central limit theorem (see for example CHANDRASEKHAR [1943], MIDDLETON [1960]) the random process $A(x, y)$ tends to a complex Gaussian process. The real and imaginary parts of the field are identically distributed with zero mean and variance $s^2/2$ and at any single point they are statistically independent. The joint probability density

function for the real and imaginary parts of the field, denoted by A_R and A_I respectively, is given by

$$p(A_R, A_I) = \frac{1}{\pi s^2} \exp\left(\frac{-(A_R^2 + A_I^2)}{s^2}\right). \tag{3}$$

Provided that the phase of the scattered field at the scatterer is uniform in the interval $-\pi$ to π and that the number N of scatterers is very large, the complex amplitude of the speckle pattern will be a complex Gaussian process regardless of whether the scatterers have a uniform or random modulus. The *convergence* of the process to a Gaussian form will in general depend on the statistics of the a_j and this is discussed further in §4.2. In many practical situations the number of scatterers is so large that problems of convergence are not important.

If the real and imaginary parts of the field have a joint Gaussian distribution, then it follows using a probability transformation that the modulus has a Rayleigh distribution, the intensity has a negative exponential distribution and the phase is uniformly distributed in the interval $-\pi$ to π:

$$\begin{aligned} p(|A|) &= \frac{2|A|}{\langle I \rangle} \exp\left(\frac{-|A|^2}{\langle I \rangle}\right) & A \geqq 0 \\ &= 0 & A < 0 \end{aligned} \tag{4}$$

$$\begin{aligned} p(I) &= \frac{1}{\langle I \rangle} \exp\left(\frac{-I}{\langle I \rangle}\right) & I \geqq 0 \\ &= 0 & I < 0 \end{aligned} \tag{5}$$

$$p(\phi) = \frac{1}{2\pi} \qquad -\pi \leqq \phi < \pi \tag{6}$$

where I is the intensity,
ϕ is the phase,
and $\langle I \rangle = s^2$ is the ensemble average (mean) intensity.

The probability density function for intensity (eq. (5)) is usually the most relevant distribution in practice; subject to the constraints on intensity of a finite mean and positivity, the negative exponential distribution has maximum entropy and indicates that a normal speckle pattern is totally random. Experimentally the probability density function for intensity $p(I)$ is found to be negative exponential to a high degree of accuracy (MCKECHNIE [1974b]), as shown in Fig. 3.

The first order statistics of speckle patterns formed in the image plane of a rough diffuser illuminated by coherent light are the same as those in the

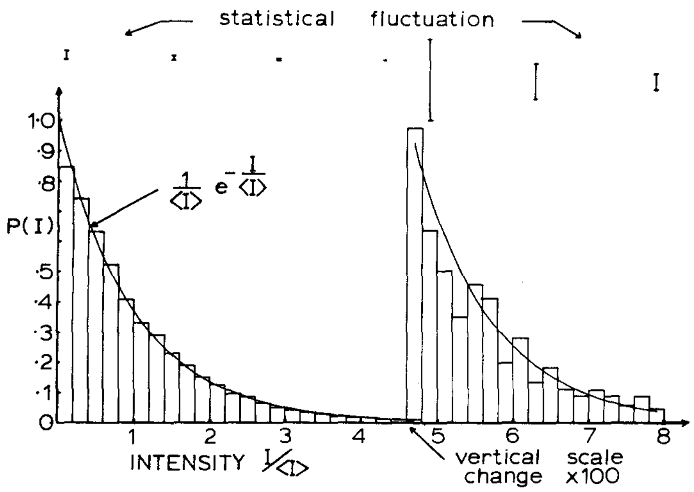

Fig. 3. A measured histogram based on some 23000 intensity measurements taken from a speckle pattern. Because the sampling aperture was small in relation to the speckle size, the histogram should have the same form as the negative exponential reference curve which is shown (McKechnie [1974b]).

Fraunhofer plane, provided that a large number of scatterers lie with the area of the point spread function of the imaging system in object space. This condition is likely to be satisfied in many practical situations, but will not hold for high resolution optical systems. The area of the point spread function of an aberration-free lens is approximately λ^2/NA^2 where NA is the numerical aperture, whereas the minimum possible phase decorrelation area of any diffuser is approximately λ^2; the maximum number of scatterers contributing to a particular image point is therefore approximately equal to $1/\mathrm{NA}^2$. This maximum number may be further reduced depending upon the actual phase decorrelation area of the diffuser used in practice. We cannot expect the statistics of the intensity to follow the negative exponential form for optical systems with numerical apertures greater than approximately 0.1. The resulting statistics for small numbers of scatterers and other surface dependent features are discussed in § 4.

2.2. SECOND ORDER STATISTICS

We again consider separately speckle patterns formed in the Fraunhofer plane and in the image plane. The second order statistics of the scattered field describe, in general terms, the spatial structure of the field. For speckle patterns formed in the Fraunhofer plane, as in Fig. 2, it is intuitively obvious

that the dimension of the finest structure in the scattered field is inversely related to the effective diameter of the scatterer (just as the dimension of the Airy disc is related to the diameter of the diffracting aperture).

The most general second order statistic of the intensity is the second order probability function, $p(I_1, I_2)$ and was first derived by von Laue [1916]. If I_1 and I_2 are the intensities at two points (x_1, y_1) and (x_2, y_2) in the Fraunhofer plane, then

$$p(I_1, I_2) = \frac{1}{\langle I\rangle^2(1-C_{12}^2)} \exp\left\{\frac{-(I_1+I_2)}{\langle I\rangle(1-C_{12}^2)}\right\} \mathscr{I}_0\left(\frac{2C_{12}\sqrt{I_1 I_2}}{\langle I\rangle(1-C_{12}^2)}\right), \qquad (7)$$

where $\mathscr{I}_0$ is the modified Bessel function of zero order and C_{12} is the modulus of the autocorrelation function of the complex amplitude and is given by

$$C_{12}(x_1-x_2, y_1-y_2) = \left|\iint_{-\infty}^{\infty} S(\xi,\eta) \exp\left\{\frac{-2\pi i}{\lambda D}\left(\xi(x_1-x_2)+\eta(y_1-y_2)\right)\right\} \mathrm{d}\xi\, \mathrm{d}\eta\right|, \qquad (8)$$

where $S(\xi,\eta)$ is the intensity distribution at the scattering plane. In the derivation of (7) and (8), we assume that a large number of randomly phased scatterers lie within the scattering aperture. It can be seen that $p(I_1, I_2)$ is completely defined in terms of the autocorrelation function C_{12}; it is a property of Gaussian processes that *all* probability density functions are completely specified by the autocorrelation function. Thus in practice the only second order statistic we need to evaluate is the autocorrelation function or its Fourier transform, the Wiener spectrum.

The autocorrelation function of the intensity in the Fraunhofer plane of a rough diffuser has been derived by many authors (von Laue [1916], Goodman [1963], Goldfischer [1965], Suzuki and Hioki [1966], Ross [1970] and Yamaguchi [1972, 1973]); here we follow Goodman's approach. The autocorrelation function of the complex amplitude $C_A(x_1, y_1)$ is defined as

$$C_A(x_1, y_1) = \langle A(x+x_1, y+y_1) A^*(x, y)\rangle, \qquad (9)$$

where $\langle\ \rangle$ denotes the ensemble average. (For a statistically stationary speckle pattern the ensemble average may be replaced by a space average.) The complex amplitude in the Fraunhofer plane of a rough scatterer is given by eq. (2), and combination of equations (2) and (9) leads to

$$C_A(x_1, y_1) = \sum_{j=1}^{N} |a_j|^2 \exp\left\{-\frac{2\pi i}{\lambda D}(\xi_j x_1 + \eta_j y_1)\right\},$$

where the various cross-products reduce to zero upon taking the average.

If the scatterers are considered to be packed sufficiently closely on the scattering area, then the scattering intensity $|a_j|^2$ can be represented by a continuous function $S(\xi, \eta)$ and we may write

$$C_A(x_1, y_1) = \int_{-\infty}^{\infty}\!\!\int S(\xi, \eta) \exp\left\{-\frac{2\pi i}{\lambda D}(x_1\xi + y_1\eta)\right\} d\xi\, d\eta. \tag{10}$$

The intensity function $S(\xi, \eta)$ is proportional to the incident intensity distribution over the scattering area and zero elsewhere, so that once this distribution is known the autocorrelation function of the complex amplitude of the speckle pattern is also known apart from a multiplicative constant.

The autocorrelation function of the intensity *fluctuation* is defined as

$$C_I(x_1, y_1) = \langle I(x+x_1, y+y_1)I(x, y)\rangle - \langle I\rangle^2. \tag{11}$$

The first term on the right-hand side of eq. (11) can be expressed in terms of $C_A(x_1, y_1)$ using a moment theorem for complex Gaussian processes due to Reed [1962]:

$$\langle Z_1^* Z_2^* Z_3 Z_4\rangle = \langle Z_1^* Z_3\rangle\langle Z_2^* Z_4\rangle + \langle Z_2^* Z_3\rangle\langle Z_1^* Z_4\rangle. \tag{12}$$

The result is

$$C_I(x_1, y_1) = |C_A(x_1, y_1)|^2 \tag{13}$$

and therefore the autocorrelation function of the intensity fluctuation in a normal speckle pattern is given by

$$C_I(x_1, y_1) = \left|\int_{-\infty}^{\infty}\!\!\int S(\xi, \eta) \exp\left\{-\frac{2\pi i}{\lambda D}(x_1\xi + y_1\eta)\right\} d\xi\, d\eta\right|^2 \tag{14}$$

where it is assumed that $S(\xi, \eta)$ is suitably normalised.

The Wiener spectrum (or power spectrum) $W(u, v)$ of the intensity fluctuation in the Fraunhofer plane, which in general terms describes the spatial frequency content of the scattered intensity, is equal to the Fourier transform of the autocorrelation function:

$$W(u, v) = \int_{-\infty}^{\infty}\!\!\int C(x_1, y_1) \exp\{-2\pi i(ux_1 + vy_1)\}\, dx_1\, dy_1$$

$$= \int_{-\infty}^{\infty}\!\!\int S(\xi, \eta) S(\xi + \lambda Du, \eta + \lambda Dv)\, d\xi\, d\eta, \tag{15}$$

since $u = \xi/\lambda D$ and $v = \eta/\lambda D$. Thus the Wiener spectrum of the intensity fluctuation is simply equal to the autocorrelation function of the *intensity* distribution across the scattering aperture suitably scaled and normalised; it should be noted that

$$\int_{-\infty}^{\infty}\!\!\int W(u, v)\, du\, dv = C(0, 0) = \sigma^2 = \langle I\rangle^2.$$

Since both the autocorrelation function and Wiener spectrum depend only on the intensity distribution across the scattering aperture, the above analysis applies also to planes other than the Fraunhofer plane, provided that a large number of scatterers contribute to the intensity at any point in the plane.

For a uniformly illuminated circular scattering area of radius r, the autocorrelation function and Wiener spectrum of the intensity fluctuation of the speckle pattern are given by

$$C(x, y) = \frac{\langle I\rangle^2 2\mathscr{J}_1((2\pi r/\lambda D)\sqrt{x^2 + y^2})}{(2\pi r/\lambda D)\sqrt{x^2 + y^2}} \tag{16}$$

and

$$W(u, v) = \frac{8\langle I\rangle^2}{\pi^2\omega_m^2}\left\{\cos^{-1}\left(\frac{\sqrt{u^2+v^2}}{\omega_m}\right) - \frac{\sqrt{u^2+v^2}}{\omega_m}\sqrt{1 + \frac{u^2+v^2}{\omega_m^2}}\right\}, \tag{17}$$

where $\mathscr{J}_1(x)$ is the first order Bessel function and ω_m is the cut-off spatial frequency given by $\omega_m = 2r/\lambda D$.

Several thousand values of the speckle pattern intensity are required to accurately verify the above results and the experimental measurements of Höhn [1968] and Dainty [1970] only approximately verified the theoretical results. However recent experimental results by McKechnie [1974a] shown in Fig. 4 indicate that the expressions given above are verified to a high degree of accuracy. The results of Fig. 4 also show that the roughness of the scatterer does not influence the Wiener spectrum (or autocorrelation function), provided of course that the illumination is perfectly coherent, that the phase fluctuation at the scatterer is uniform in the interval $-\pi$ to π and that a large number of scatterers contribute to the intensity at any point in the Fraunhofer plane.

The autocorrelation function and Wiener spectrum of speckle patterns produced in the image plane of rough diffusers may be found either by using an extension of the Fraunhofer case (Enloe [1967]) or by straightforward application of linear filter and square law detection theory (Burck-

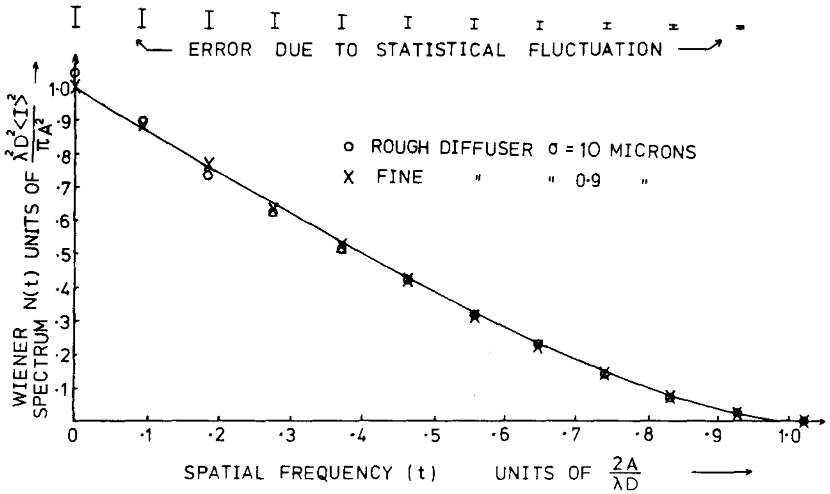

Fig. 4. Measured Wiener spectra of the intensity of speckle patterns observed in the Fraunhofer plane of two optically rough diffusers. The solid line is the form predicted using eq. (17) (McKechnie [1974a]).

HARDT [1970], DAINTY [1970], LOWENTHAL and ARSENHAULT [1970]). The latter approach is described here.

Let the optical system be represented by a linear filter whose isoplanatic amplitude point spread function is $P(x, y)$ and transfer function is $T(u, v)$; the transfer function is equal to the pupil function $H(\xi, \eta)$ suitably scaled,

$$T(u, v) = H(\lambda f u, \lambda f v), \tag{18}$$

where f is the focal length. If the complex amplitude distribution of an object is $A(x_1, y_1)$, then the complex amplitude distribution in the image $A'(x, y)$ is given by

$$A'(x, y) = \int\int_{-\infty}^{\infty} A(x_1, y_1) P(x - x_1, y - y_1) \, dx_1 \, dy_1 .$$

Suppose that the object distribution $A(x_1, y_1)$ is a stationary statistical process with a Wiener spectrum $W_A(u, v)$. The spectrum of the output amplitude of the linear filter is given by a well-known result (MIDDLETON [1960]),

$$W_A'(u, v) = |T(u, v)|^2 W_A(u, v). \tag{19}$$

The Wiener spectrum of the intensity in the speckle pattern $W(u, v)$ is found from the Wiener spectrum of the complex amplitude $W_A'(u, v)$ by

applying a result derived by RICE [1954] in his analysis of the square law detection problem:

$$W(u, v) = \int\int_{-\infty}^{\infty} W_A'(u_1, v_1) W_A'(u_1 + u, v_1 + v) \, du_1 \, dv_1 . \tag{20}$$

If the diffuser structure is very much finer than the diameter of the point spread function (this condition is also necessary for Gaussian statistics for the image amplitude), the Wiener spectrum of the object amplitude $W_A(u, v)$ will be constant for the values of (u, v) for which $|T(u, v)|^2$ is significantly non-zero, and combination of equations (18) to (20) gives, for this special case of a "white" noise object,

$$W(u, v) = k \int\int_{-\infty}^{\infty} |H(\xi, \eta)|^2 |H(\xi + \lambda f u, \eta + \lambda f v)|^2 \, d\xi \, d\eta, \tag{21}$$

where k is a normalisation constant.
Comparison of eq. (15) for the Fraunhofer plane and eq. (21) for the image plane shows that the Wiener spectra have the same form in each case. For speckle patterns produced in the Fraunhofer plane the intensity distribution of the illumination is the quantity that determines the spectrum, whereas in the image plane it is the squared modulus of the pupil function.

For an unshaded circular imaging pupil of radius r, the autocorrelation function and Wiener spectrum of the intensity fluctuation are given by equations (16) and (17) respectively, with the distance D replaced by the focal length f. The autocorrelation function has the same form as the intensity distribution of an Airy disc, and this fact has given rise to the general rule-of-thumb that the speckle "size" is of the same order of magnitude as the size of the Airy disc; note however that the autocorrelation function and hence the speckle "size" is independent of any aberration of the imaging system. When imaging a general diffuse object, the object amplitude is not statistically stationary, and the statistics of the image are described by non-stationary functions; this is discussed by LOWENTHAL and ARSENHAULT [1970] as an extension of the above arguments for the stationary case.

2.3. STATISTICS OF THE MEASURED INTENSITY

The statistical properties we have evaluated so far all relate to the complex amplitude or intensity at one or more *points* in a speckle pattern. In prac-

tice, speckle patterns are more likely to be averaged over some non-zero area by, for example, a scanning aperture and we shall call this averaged or integrated intensity the *measured* intensity. The measured intensity $I'(x,y)$ is related to the intensity of the speckle pattern $I(x_1, y_1)$ by a convolution formula,

$$I'(x, y) = \int_{-\infty}^{\infty}\int I(x_1, y_1)B(x-x_1, y-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1, \tag{22}$$

where $B(x,y)$ is the intensity point spread function of the averaging device normalised such that the volume of $B(x,y)$ is unity.

The Wiener spectrum of the measured intensity fluctuation $W'(u,v)$ is simply related to that of the speckle pattern intensity:

$$W'(u, v) = W(u, v)|b(u, v)|^2, \tag{23}$$

where $b(u,v)$ is the Fourier transform of $B(x,y)$. The variance of the measured intensity σ_b^2 is given by

$$\sigma_b^2 = \int_{-\infty}^{\infty}\int W'(u, v)\,\mathrm{d}u\,\mathrm{d}v = \int_{-\infty}^{\infty}\int W(u, v)|b(u, v)|^2\,\mathrm{d}u\,\mathrm{d}v. \tag{24}$$

The Wiener spectra of the intensity fluctuation in normal speckle patterns formed in the Fraunhofer and image planes are given by equations (15) and (21) respectively.

GERRITSON, HANNAN and RAMBERG [1968] calculated the ratio $\langle I\rangle/\sigma_b$ (a "signal-to-noise" ratio) for square pupils and scanning apertures. For a circular imaging pupil of radius r and a circular scanning aperture of radius a, eq. (24) becomes,

$$\frac{\sigma_b^2}{\langle I\rangle^2} \simeq \frac{43}{\pi^3k^2}\int_0^1 (\cos^{-1}(\omega)-\omega\sqrt{1-\omega^2})\,\frac{\mathscr{J}_1(1.22\pi k\omega)}{\omega}\,\mathrm{d}\omega \tag{25}$$

where

$$k = \frac{\text{diameter of scanning aperture}}{\text{Rayleigh resolution limit of lens}} = \frac{2a}{0.61\lambda f/r}.$$

The quantity k is a measure of the aperture size relative to the speckle "size", since we showed in § 2.2 that the speckle "size" is of the same order of magnitude as the Airy disc. In Fig. 5, $\sigma_b/\langle I\rangle$ is plotted as a function of the relative aperture diameter for both top-hat and Gaussian scanning apertures.

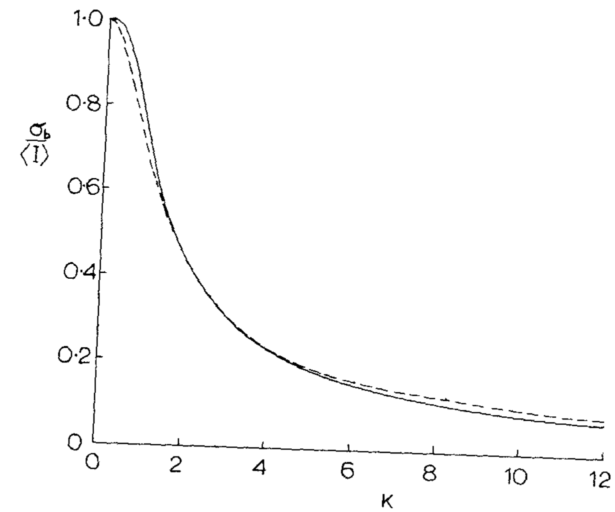

Fig. 5 The standard deviation of the intensity fluctuations relative to the mean intensity for top-hat and Gaussian scanning apertures for a speckle pattern formed by an optical system with a top-hat pupil function. For the top-hat aperture, ———, k = (diameter of scanning aperture)/(Rayleigh resolution limit of lens). For the Gaussian aperture, --------, k = (diameter of aperture at $1/e$ points)/(Rayleigh resolution limit of lens).

If the scanning aperture is very large compared to the speckle "size" then the variance is given approximately by

$$\sigma_b^2 \simeq W(0, 0)\int_{-\infty}^{\infty}\int |b(u, v)|^2\,\mathrm{d}u\,\mathrm{d}v.$$

For unshaded pupils and scanning apertures, this expression reduces to

$$\frac{\sigma_b^2}{\langle I\rangle^2} \simeq \frac{1}{\mathscr{A}_b\mathscr{A}_p}, \tag{26}$$

where $\mathscr{A}_b$ is the area of the scanning or averaging aperture, and $\mathscr{A}_p$ is the area of the pupil in spatial frequency units (i.e. $\mathscr{A}_p = \pi r^2/\lambda^2 f^2$).

Finding the first order probability density function for the measured intensity, $p(I')$, is not straightforward. The problem is analogous to that of square-law detection and low-pass filtration in the one-dimensional time domain, which was analysed by KAC and SIEGERT [1947] and SLEPIAN [1958]. The extension* to the case of the measured intensity in a speckle

* All three authors in fact made errors in extending the results of KAC and SIEGERT [1947]; these arose as a result of failing to include all aspects of the fact that the amplitude is a *complex* Gaussian process.

pattern was made by CONDIE [1966], DAINTY [1971] and BARAKAT [1973].

Equation (22) for the measured intensity may be rewritten in terms of the complex amplitude in the speckle pattern $A(x_1, y_1)$ as,

$$I'(x, y) = \iint_{-\infty}^{\infty} |A(x_1, y_1)|^2 B(x-x_1, y-y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1 .$$

The problem in finding $p(I')$ is that I' is a weighted sum of *correlated* random variables. By expanding $A(x_1, y_1)$ in terms of orthogonal functions (the Karhunen–Loève expansion), the measured intensity can be expressed as a weighted sum of *independent* random variables and following this approach the probability density function for the measured intensity is given by

$$p(I') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}zI')}{\prod_n (1-\mathrm{i}z\varepsilon_n)}\,\mathrm{d}z, \tag{27}$$

where ε_n are the eigenvalues of the homogenous Fredholm equation

$$\varepsilon_n \psi_n(x, y) = \iint_{-\infty}^{\infty} C_A(x-x_1, y-y_1) B(x_1, y_1) \psi_n(x_1, y_1)\,\mathrm{d}x_1\,\mathrm{d}y_1, \tag{28}$$

with $\varepsilon_0 \geqq \varepsilon_1 \geqq \varepsilon_2 \ldots$ and $C_A(x, y)$ is the autocorrelation function of the complex amplitude in the speckle pattern.

It is in general difficult to calculate the eigenvalues from eq. (28); however, if the eigenvalues ε_n are plotted as a function of n it is often found that for a wide range of $C_A(x_1, y_1)$ and $B(x_1, y_1)$, the eigenvalues are approximately equal to ε_0 up to a value of $n \simeq n_0$, and approximately equal to zero for $n > n_0$. With this approximation, the probability density function for the measured intensity becomes (CONDIE [1966], SCRIBOT [1974]),

$$p(I') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-\mathrm{i}zI')}{(1-\mathrm{i}z\varepsilon_0)^n}\,\mathrm{d}z,$$

which, upon integration, gives the gamma distribution,

$$p(I') = \left(\frac{1}{\varepsilon_0}\right)^{n_0} \frac{I'^{n_0-1}\exp(-I'/\varepsilon_0)}{\Gamma(n_0)}. \tag{29}$$

It can also be shown that, in this approximate case,

$$\langle I \rangle = \varepsilon_0 n_0$$

and

$$\sigma_b^2 = \varepsilon_0^2 n_0 .$$

Equation (29) can be therefore rewritten as

$$p(I') = \left(\frac{n_0}{\langle I \rangle}\right)^{n_0} \frac{I'^{n_0-1}\exp(-I'n_0/\langle I \rangle)}{\Gamma(n_0)}, \tag{30}$$

where n_0 is interpreted as the number of independent correlation cells (speckles) within the scanning aperture.

It is interesting to note that this approximate expression for the probability density function of the measured intensity has also been derived using a heuristic argument (GOODMAN [1965]) that is again an extension of work in the one-dimensional time domain (RICE [1954], MANDEL [1959]). The scanning aperture is regarded as consisting of n_0 independent correlation cells, the intensity being taken as constant within any one cell and statistically independent of the intensity in all other cells. The intensity of each correlation cell is assumed to have a negative exponential distribution and therefore the total measured intensity is approximately a gamma variate, as in eq. (30), where n_0 is chosen such that the variance of the approximate and exact distributions are equal:

$$n_0 = \langle I \rangle^2/\sigma_b^2. \tag{31}$$

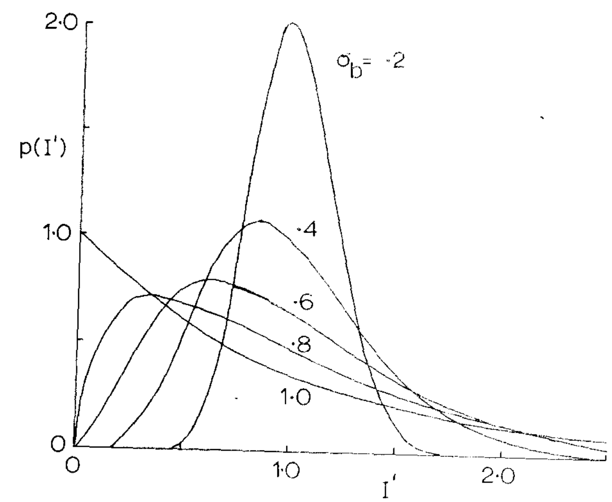

Fig. 6. The approximate probability density function of a normal speckle pattern scanned by an aperture. This distribution has two parameters, the mean intensity (assumed to be unity) and the standard deviation σ_b of the intensity fluctuation (relative to the mean intensity).

In Fig. 6 the approximate probability density function is drawn for a mean measured intensity of unity and various values of $\sigma_b/\langle I\rangle$; for $\sigma_b/\langle I\rangle \to 1$, the distribution tends to the negative exponential form and for $\sigma_b/\langle I\rangle \to 0$ it tends to a Gaussian form.

An important difference between the approximate distribution given by eq. (30) and the exact distribution given by eqs. (27) and (28) is that the approximate distribution depends on the Wiener spectrum of the speckle only through a weighted integral (eq. (24)) whereas the exact distribution depends on the *form* of the Wiener spectrum. It is found however that the agreement between the approximate and exact formulae is close regardless of the form of the speckle Wiener spectrum (MANDEL [1959], BEDARD, CHANG and MANDEL [1967], SCRIBOT [1974]).

§ 3. Partially Coherent Illumination

3.1. SPATIAL COHERENCE – FRAUNHOFER PLANE

Speckle patterns were first observed in situations where the light incident on the scatterer was both spatially and temporally partially coherent, and the properties of such patterns are somewhat less straightforward than those for perfectly coherent illumination. For each case of spatial and temporal partial coherence we again make the subdivision into speckle patterns formed in the region of the Fraunhofer plane and those formed in the image plane, although a perfectly general theory may of course be used to apply to both cases (PARRY [1975b]).

In Fig. 7 we show an optical system that might be used to form a speckle pattern in the Fraunhofer plane of a diffuser illuminated by spatially partially coherent light. The diffuser is uniformly illuminated by light which has a mutual coherence function $\Gamma(\xi_1, \xi_2, \eta_1, \eta_2)$ given by the

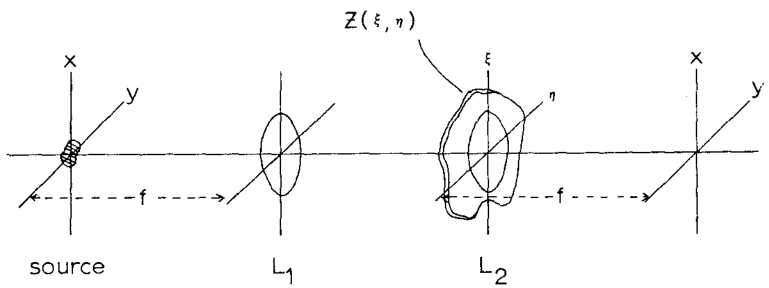

Fig. 7. Formation of a speckle pattern produced in the Fraunhofer plane of a diffuser illuminated by spatially partially coherent light.

inverse Fourier transform of the source intensity $s(x, y)$ suitably scaled and normalised. The area of the diffuser that contributes to the speckle pattern is limited by the extent of the lens L_2 whose intensity transmittance is equal to the squared modulus of the pupil function, $|H(\xi, \eta)|^2$.

To find the statistics of the speckle pattern intensity $I(x, y)$ formed in this case, we must first obtain a relationship between $I(x, y)$, $s(x, y)$, $H(\xi, \eta)$ and $Z(\xi, \eta)$, where this last function is the complex amplitude at the pupil due to the diffuser. There are two conceptually different approaches that have been used to obtain this relationship. The first (LABEYRIE [1970], DAINTY [1973]) is based on the fact that $I(x, y)$ is an image, albeit a very poor one, of the source intensity and the usual equations for imaging of self-luminous (incoherent) objects apply; the effective pupil function is equal to the product $Z(\xi, \eta)\, H(\xi, \eta)$. This approach is useful when the diffuser is weak, as in the case of the atmosphere; the speckle pattern only exists in a small region of the x-plane (see § 4.2). The second, more conventional, approach is to follow the mutual coherence function as it propagates through the system (ROSS [1969], ASAKURA, FUJII and MURATA [1972], FUJII and ASAKURA [1973, 1974a]) and this is outlined below.

Assuming that the illumination is quasimonochromatic and that the mutual coherence function is stationary, then (BORN and WOLF [1970]),

$$I(x, y) = \iint_{-\infty}^{\infty} \Gamma(\xi_1 - \xi_2, \eta_1 - \eta_2) H(\xi_1, \eta) Z(\xi_1, \eta) H^*(\xi_2, \eta_2) Z^*(\xi_2, \eta_2)$$
$$\times \exp\left\{\frac{-2\pi i}{\lambda f}\left[x(\xi_1 - \xi_2) + y(\eta_1 - \eta_2)\right]\right\} d\xi_1\, d\eta_1\, d\xi_2\, d\eta_2. \tag{32}$$

Making the substitutions

$$u = \frac{\xi_1 - \xi_2}{\lambda f}, \quad v = \frac{\eta_1 - \eta_2}{\lambda f}$$

we obtain

$$I(x, y) = \iint_{-\infty}^{\infty} \Gamma(\lambda f u, \lambda f v) \left\{ \iint_{-\infty}^{\infty} H(\lambda f u + \xi_2, \lambda f v + \eta_2) \right.$$
$$\left. \times Z(\lambda f u + \xi_2, \lambda f v + \eta_2) H^*(\xi_2, \eta_2) Z^*(\xi_2, \eta_2)\, d\xi_2\, d\eta_2 \right\}$$
$$\times \exp\{-2\pi i(ux + vy)\}\, du\, dv$$

which further simplifies to

$$I(x, y) = s(x, y) \circledast \left| \int\int_{-\infty}^{\infty} H(\lambda f u, \lambda f v) Z(\lambda f u, \lambda f v) \exp\{-2\pi i(ux+vy)\} \, du \, dv \right|^2,$$

where $\circledast$ denotes convolution. However the second term in the convolution is simply the normal speckle pattern that would be obtained in perfectly coherent light (provided of course that the complex amplitude transmittance of the diffuser $Z(x, y)$ satisfies the constraints given in § 2). Thus,

$$I(x, y) = s(x, y) \circledast I_c(x, y), \tag{33}$$

where $I_c(x, y)$ is the speckle pattern intensity that would be obtained with perfectly coherent illumination.

Equation (33) relates the intensities of speckle patterns produced in the Fraunhofer plane in spatially partially coherent and fully coherent illumination, and the first and second order statistics of the partially coherent case follow quite simply. The effect of the non-zero source extent $s(x, y)$ is to blur out and to reduce the contrast of the fully coherent speckle pattern. Comparison of eq. (33) with eq. (22), shows that the source distribution $s(x, y)$ in the partially coherent case plays the same role as the scanning aperture $B(x, y)$ in the case of the *measured* intensity in coherent light. Thus the exact first order probability density function is given by eqs. (27) and (28) with $s(x, y)$ substituted for $B(x, y)$, or approximately by

$$p(I) = \left(\frac{n_0}{\langle I \rangle}\right)^{n_0} \frac{I^{n_0-1} \exp[-In_0/\langle I \rangle]}{\Gamma(n_0)}, \tag{30}$$

where n_0 is the number of (coherent) speckles that lie with the source function $s(x, y)$.

The Wiener spectrum of the intensity fluctuation in a speckle pattern formed by spatially partially coherent illumination of a diffuser is given by

$$W(u, v) = W_c(u, v)|\Gamma(\lambda f u, \lambda f v)|^2 \tag{34}$$

where $\Gamma(\xi, \eta)$ is the mutual coherence function and $W_c(u, v)$ is the Wiener spectrum of a coherent speckle pattern. Equation (34) is the basis of a method of measuring the mutual coherence function; providing the diffuser satisfies the conditions laid down in § 2, the function $W_c(u, v)$ is of a known form (eq. (21) in the present notation) and by measuring $W(u, v)$ the squared modulus of $\Gamma(\xi, \eta)$ can be found. An example of an experimental result is given in Fig. 8; the same basic method is also used in astronomy for the determination of stellar diameters and binary star separations.

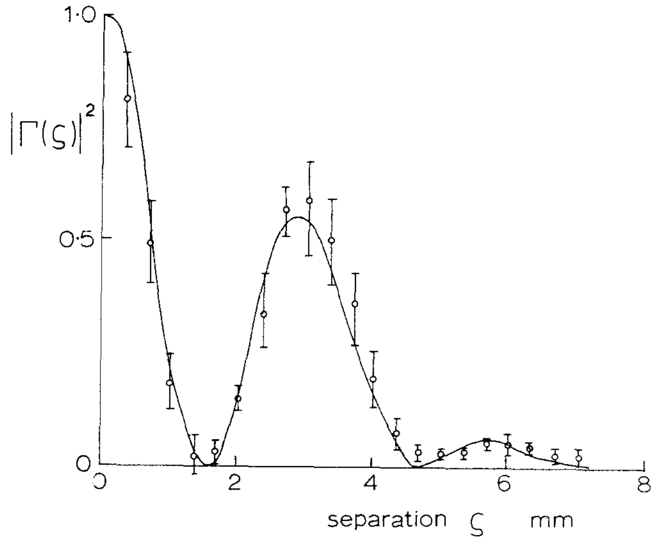

Fig. 8. An experimental measurement of the spatial coherence of a light source using speckle patterns (FUJII and ASAKURA [1973]).

3.2. SPATIAL COHERENCE – IMAGE PLANE

The statistics of the intensity in the image of a spatially partially coherently illuminated diffuser are slightly more difficult to find than those in the Fraunhofer case. Referring to Fig. 9, the image intensity is given by

$$I(x, y) = \int\int\int\int_{-\infty}^{\infty} \Gamma(x_1 - x_2, y_1 - y_2) h(x - x_1, y - y_1)$$
$$\times h^*(x - x_2, y - y_2) Z(x_1, y_1) Z^*(x_2, y_2) \, dx_1 \, dy_1 \, dx_2 \, dy_2, \tag{35}$$

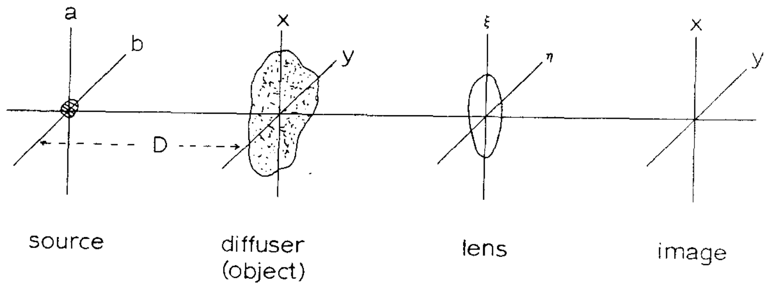

Fig. 9. Formation of a speckle pattern produced in the image plane of a diffuser illuminated by spatially partially coherent light.

where $h(x, y)$ is the amplitude point spread function of the imaging system and $Z(x, y)$ is the complex amplitude transmittance of the diffuser. In terms of the effective source $J(u, v) = s(a/\lambda D, b/\lambda D)$,

$$J(u, v) = \iint_{-\infty}^{\infty} \Gamma(x, y) \exp\{-2\pi i(ux+vy)\}\, dx\, dy,$$

this may be written as

$$I(x, y) = \iint_{-\infty}^{\infty} J(u, v)|Q(x, y; u, v)|^2\, du\, dv, \tag{36}$$

where $Q(x, y, u, v)$ is the amplitude in a speckle pattern produced by an oblique coherent wave incident on the diffuser at angle $(\lambda u, \lambda v)$,

$$Q(x, y; u, v) = \iint_{-\infty}^{\infty} h(x-x_1, y-y_1) Z(x_1, y_1) \exp\{-2\pi i(ux_1+vy_1)\}\, dx_1\, dy_1.$$

Equation (36) states that the speckle pattern produced in the image plane in spatially partially coherent illumination is just the (weighted) sum of the intensities of speckle patterns produced by different angles of coherent illumination. The exact first order probability density function is therefore given by an expression similar to eq. (27), where the eigenvalues ε_n are given by an expression similar to (28). The approximate probability density function is given by equation (30), where the value of n_0 is approximately equal to the ratio of the diameter of the point spread function to the diameter of the coherence patch at the diffuser. In an optical system which has an illuminating condenser of numerical aperture $\mathrm{NA_c}$ and an imaging objective numerical aperture of $\mathrm{NA_0}$ then $n_0 \simeq \mathscr{S} = \mathrm{NA_c}/\mathrm{NA_0}$ (for $n_0 \gg 1$).

The Wiener spectrum of the intensity fluctuation in a speckle pattern produced in a spatially partially coherent optical system has been derived for optically very rough surfaces by Dainty [1970] and Yamaguchi [1973], and for surfaces of an arbitary magnitude of roughness but with a Gaussian phase profile by Fujii and Asakura [1974c]; this latter case is discussed in § 4.3, and only the result for a very rough surface is found here.

Equation (35) for the image intensity may be rewritten in the form

$$I(x) = \iint_{-\infty}^{\infty} V(x_1, x_2) Z(x-x_1) Z^*(x-x_2)\, dx_1\, dx_2,$$

where

$$V(x_1, x_2) = \Gamma(x_1-x_2) h(x_1) h^*(x_2).$$

One dimensional notation is used here to simplify the derivation. The autocorrelation of the intensity fluctuation $I'(x) = I(x) - \langle I \rangle$ is given by

$$\begin{aligned} C(x_0) &= \langle I'(x) I'(x+x_0) \rangle \\ &= \iint_{-\infty}^{\infty} V(x_1, x_2) V(x_3, x_4) \\ &\quad \times \{\langle Z(x-x_1) Z^*(x-x_2) Z(x+x_0-x_3) Z^*(x+x_0-x_4) \rangle \\ &\quad - \langle Z(x-x_1) Z^*(x-x_2) \rangle \langle Z(x+x_0-x_3) Z^*(x+x_0-x_4) \rangle\} \\ &\quad \times dx_1\, dx_2\, dx_3\, dx_4. \end{aligned} \tag{37}$$

It is now assumed that the object amplitude $Z(x)$ is a complex Gaussian process so that applying the theorem of Reed [1962] to the fourth order moment we obtain,

$$\begin{aligned} C(x_0) = \iiiint_{-\infty}^{\infty} & V(x_1, x_2) V(x_3, x_4) \{\langle Z(x-x_1) Z^*(x+x_0-x_4) \rangle \\ & \times \langle Z^*(x-x_2) Z(x+x_0-x_3) \rangle\}\, dx_1\, dx_2\, dx_3\, dx_4, \end{aligned}$$

or in terms of the autocorrelation of the diffuser complex amplitude $C_z(x)$,

$$\begin{aligned} C(x_0) = \iint_{-\infty}^{\infty} & V(x_1, x_2) V(x_3, x_4) \{C_z^*(x_1-x_4+x_0) \\ & \times C_z(x_2-x_3+x_0)\}\, dx_1\, dx_2\, dx_3\, dx_4. \end{aligned} \tag{38}$$

The Wiener spectrum of the intensity fluctuations is the Fourier transform of eq. (38), and reduces to

$$W(u) = \int_{-\infty}^{\infty} W_z(u_1) W_z(u+u_1) \mathscr{T}(-u_1, -(u+u_1)) \mathscr{T}(u+u_1, u_1)\, du_1, \tag{39}$$

where $W_z(u)$ is the Wiener spectrum of the object amplitude, and $\mathscr{T}(u_1, u_2)$ is the transmission cross-coefficient (Born and Wolf [1970]) and is given by

$$\mathscr{T}(u_1, u_2) = \int_{-\infty}^{\infty} J(u) H(u+u_1) H^*(u-u_2) \mathrm{d}u_1 \, \mathrm{d}u_2 .$$

Since $\mathscr{T}(u_1, u_2) = \mathscr{T}^*(-u_2, -u_1)$, eq. (39) reduces further to

$$W(u) = \int_{-\infty}^{\infty} W_z(u_1) W_z(u+u_1) |\mathscr{T}(u+u_1, u_1)|^2 \mathrm{d}u_1 \tag{40}$$

and for a diffuser whose lateral structure is small compared with the point spread function of the imaging system (a "white-noise" diffuser),

$$W(u) = \int_{-\infty}^{\infty} |\mathscr{T}(u+u_1, u_1)|^2 \mathrm{d}u_1 . \tag{41}$$

It should be noted that the result of eq. (41) for a white noise diffuser is *not* restricted to the case of Gaussian statistics for the diffuser complex amplitude transmittance; if the function $Z(x, y)$ has an autocorrelation function that is a delta-function (i.e. white noise), then it can be shown that eq. (41) must follow regardless of the statistics of $Z(x, y)$.

In Fig. 10, the scale value $W(0)$ of the Wiener spectrum is plotted as a function of the coherence parameter $\mathscr{S}$

$$\mathscr{S} = \frac{\text{radius of effective source}}{\text{radius of entrance pupil}} = \frac{\mathrm{NA_c}}{\mathrm{NA_0}} .$$

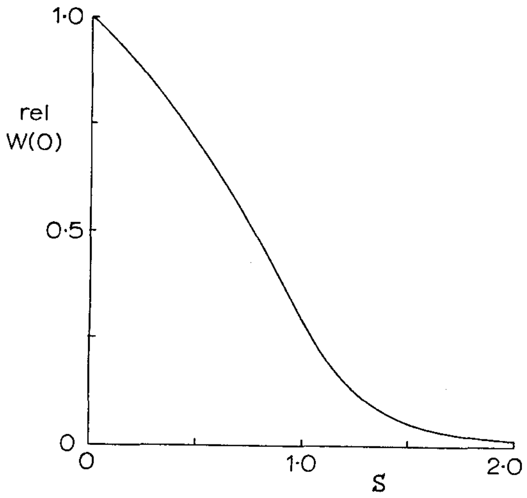

Fig. 10. The zero spatial frequency value of the Wiener spectrum of the intensity fluctuation in a speckle pattern formed by an aberration-free partially coherent optical system with a top-hat effective source distribution, plotted as a function of the coherence parameter $\mathscr{S}$ (DAINTY [1970]).

The quantity $W(0)$ gives an indication of the variance σ^2 of the speckle pattern, and it is clear from the figure that the speckle contrast remains high for $\mathscr{S} < 1$. This result is also confirmed by FUJII and ASAKURA [1974c], and is discussed further in § 4.3.

3.3. TEMPORAL COHERENCE

Much of the early discussion on the nature of speckle patterns was concerned with the effect of the non-monochromaticity of the light source (DE HAAS [1918], RAMACHANDRAN [1943]). Speckle patterns formed in the Fraunhofer plane of diffusers illuminated by all lines of an argon laser are shown in Fig. 11 for three surfaces of different roughness (a colour plate is given by MARTIENSSEN and SPILLER [1965]). Speckle patterns formed by non-monochromatic sources have two notable features that distinguish them from normal (monochromatic) speckle patterns. Firstly there is a radial structure which depends both on the temporal coherence of the source and on the roughness of the diffuser; secondly, the contrast of the pattern is a function of position in the Fraunhofer plane and also depends on the temporal coherence and surface roughness.

For a surface whose r.m.s. height variation σ_h is greater than one wavelength but less than the coherence length L_c of the incident radiation (see Fig. 12 (a)), each wavelength forms a normal speckle pattern that is fairly well correlated with the speckle patterns produced by neighbouring wavelengths, at least near the centre of the pattern. The speckle size is of course related directly to the wavelength and the overall effect is to produce a radial structure in the total pattern. The "length" of a speckle in the radial structure depends upon the magnitude of the relative phase changes from point to point on the diffuser as a function of angle and this depends on the r.m.s. height variation σ_h; for a large surface roughness the relative phase change is large for a small change in angle and the radial structure is less apparent (see Fig. 11). The contrast of the pattern is also less for surfaces whose r.m.s. roughness is greater than the coherence length of the incident light (see Fig. 12 (b)) because at any point in the Fraunhofer plane fewer scatterers give coherent contributions and more give incoherent contributions. For normal incidence and observation at the centre of the Fraunhofer plane, GOODMAN [1963] showed that high contrast is only achieved if $\sigma_h < L_c$.

More complete treatments of the statistics of speckle patterns formed in the Fraunhofer plane in temporally partially coherent ("white") light have been given by GOODMAN [1963], FUJII and ASAKURA [1974a], PARRY

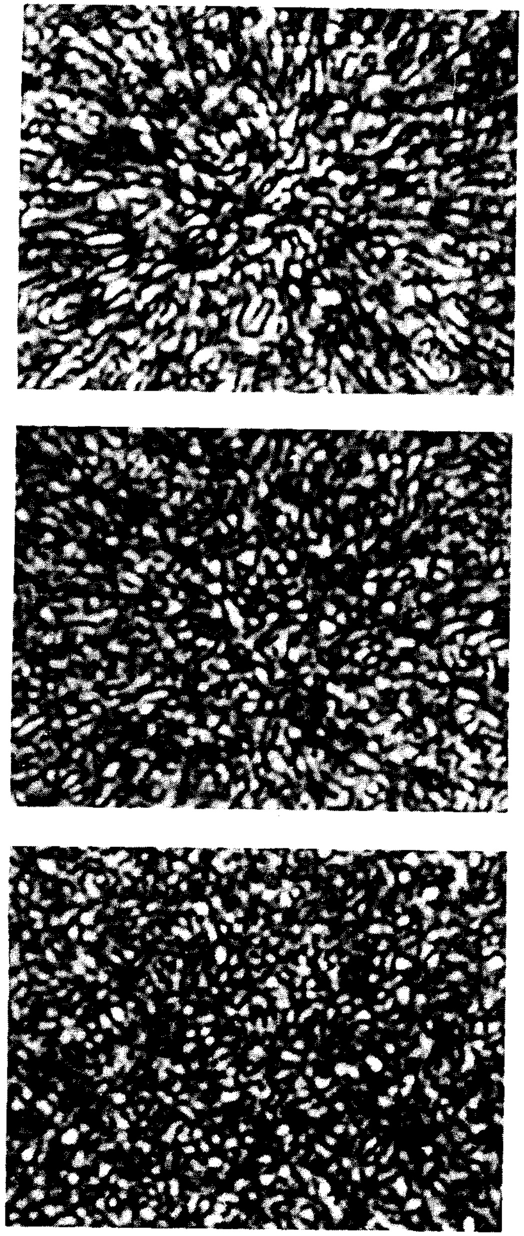

Fig. 11. Speckle patterns produced in the Fraunhofer planes of diffusers with three surface roughnesses illuminated by an argon laser (wavelengths present 514, 496, 488, 476 nm): (a) $\sigma_h \simeq 1\mu$m, (b) $\sigma_h \simeq 3\mu$m, (c) $\sigma_h \simeq 10\mu$m (PARRY [1974b]).

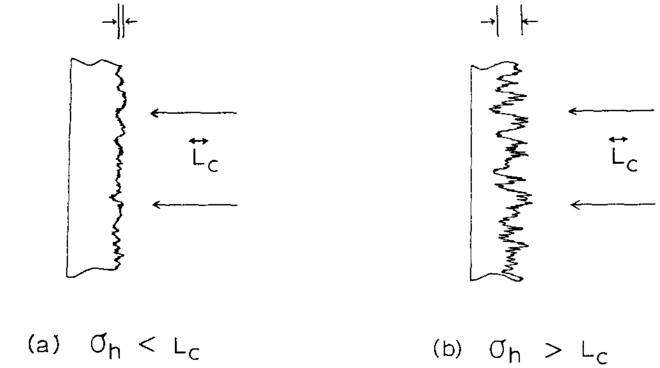

Fig. 12. Surfaces whose r.m.s. height variation σ_h are less and greater than the coherence length L_c.

[1974a, b, 1975a, b] and PEDERSEN [1975a, b]. The results given so far in this article have applied to all surfaces whose r.m.s. roughness is greater than one wavelength, regardless of the detailed statistical properties of the surface. However we can no longer accept this naïve picture of the scattering surface and it is necessary to introduce surface dependent parameters using the Beckmann model (BECKMANN and SPIZZICHINO [1963]).

We consider the scattering geometry shown in Fig. 13 (PEDERSEN [1975b]). The surface lies in the $(x, y) \equiv \boldsymbol{x}$ plane and has a profile $h(\boldsymbol{x})$ whose r.m.s. fluctuation is $\sigma_h > \lambda$. The surface is illuminated by a unit amplitude plane wave with wave vector $\boldsymbol{k}_0$, and the scattered wave $\boldsymbol{k}$ is observed in the Fraunhofer plane, where $|\boldsymbol{k}| = |\boldsymbol{k}_0| = 2\pi/\lambda$. The scattered amplitude is a

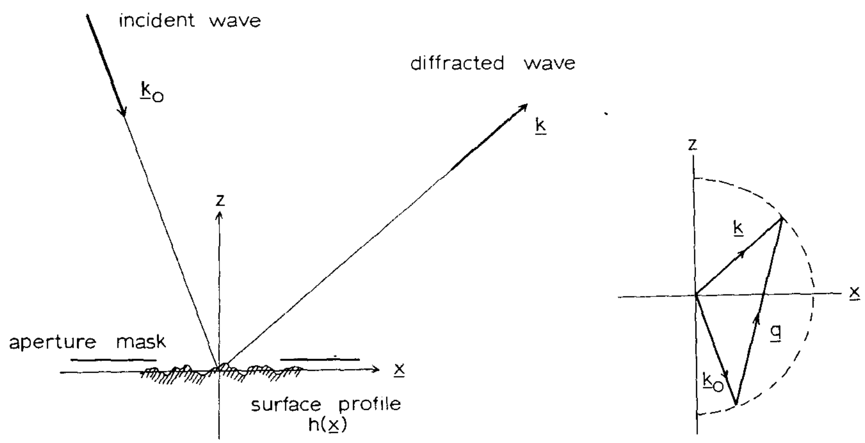

Fig. 13. Diffraction geometry for the formation of speckle patterns (PEDERSEN [1975b]).

function of $\boldsymbol{k}_0$ and $\boldsymbol{k}$ only through the difference $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}_0$ as multiple reflections are ignored. The problem of finding the statistics of the scattered field for surfaces of arbitrary profile $h(\boldsymbol{x})$ is difficult. If the surface roughness is greater than one wavelength and we illuminate the surface with monochromatic light then as we have seen in § 2 it is possible to obtain a useful statistical description of the scattered field. However, it should be noted that the description given in § 2 does not explain the roughness dependent angular correlation of speckle patterns (ARCHBOLD and ENNOS [1972]).

In this present case we make two important assumptions about the surface profile. Firstly we assume that $h(\boldsymbol{x})$ is a stationary Gaussian random variable with zero mean and autocorrelation function $C_h(\boldsymbol{x}_1 - \boldsymbol{x}_2)$; the first order characteristic function $\Phi_1(t)$ is therefore given by

$$\Phi_1(t) = \langle \exp(ith) \rangle = \exp(-\tfrac{1}{2}\sigma_h^2 t^2). \tag{42}$$

Secondly it is assumed that the total area of the illuminated surface is large compared to the area of correlated surface structure (i.e. large compared to the extent of $C_h(\boldsymbol{x}_1 - \boldsymbol{x}_2)$) so that there are many correlation areas within the illuminated area. Provided that these two assumptions are made, it can be shown that (regardless of the form of $C_h(\boldsymbol{x}_1 - \boldsymbol{x}_2)$) the normalised autocorrelation function of the scattered intensity is locally stationary and given by

$$C(\Delta \boldsymbol{q}) = \Phi_1^2(\Delta q_z) C(\Delta \boldsymbol{q}_x), \tag{43}$$

where $\Delta \boldsymbol{q} = \boldsymbol{q}_1 - \boldsymbol{q}_2$, Δq_z is the projection of $\Delta \boldsymbol{q}$ on the z axis, and $\Delta \boldsymbol{q}_x$ is the projection of $\Delta \boldsymbol{q}$ on the $\boldsymbol{x}$ plane. $C(\Delta \boldsymbol{q}_x)$ is the normalised autocorrelation function of the intensity in a "normal" speckle pattern which in the present notation is given by

$$C(\Delta \boldsymbol{q}_x) = \left| \frac{\displaystyle\iint_{-\infty}^{\infty} S(\boldsymbol{x}) \exp(-\mathrm{i}\Delta \boldsymbol{q}_x \cdot \boldsymbol{x})\, \mathrm{d}^2 \boldsymbol{x}}{\displaystyle\iint_{-\infty}^{\infty} S(\boldsymbol{x})\, \mathrm{d}^2 \boldsymbol{x}} \right|^2, \tag{44}$$

where $S(\boldsymbol{x})$ is the intensity distribution of the incident light.

It is useful at this point to let

$$\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}_0 = k(\boldsymbol{n} - \boldsymbol{n}_0) = k\boldsymbol{m}$$

where k is the wavenumber ($k = 2\pi/\lambda$) and $\boldsymbol{m} = \boldsymbol{n} - \boldsymbol{n}_0$ is the change in propagation direction between the incident and scattered wave.

If a polychromatic speckle pattern is formed using light whose spectral intensity distribution is $S(k)$, the intensity of the observed speckle pattern $I(\boldsymbol{m})$ is simply the sum of the monochromatic patterns $I(k\boldsymbol{m})$ suitably weighted by $S(k)$:

$$I(\boldsymbol{m}) = \int_{-\infty}^{\infty} S(k) I(k\boldsymbol{m})\, \mathrm{d}k.$$

It follows that the mean intensity $\langle I(\boldsymbol{m}) \rangle$ is given by

$$\langle I(\boldsymbol{m}) \rangle = \int_{-\infty}^{\infty} S(k) \langle I(k\boldsymbol{m}) \rangle\, \mathrm{d}k \tag{45}$$

and provided that this mean is locally stationary over regions of speckle correlation, the normalised angular autocorrelation function $C_{\mathrm{w}}(\boldsymbol{m}_1, \boldsymbol{m}_2)$ of the white light speckle intensity fluctuation is given by

$$C_{\mathrm{w}}(\boldsymbol{m}_1, \boldsymbol{m}_2) = \iint_{-\infty}^{\infty} S(k_1) S(k_2) C(k_2 \boldsymbol{m}_2 - k_1 \boldsymbol{m}_1)\, \mathrm{d}k_1\, \mathrm{d}k_2, \tag{46}$$

where $S(k)$ is normalised so that $\int_{-\infty}^{\infty} S(k)\, \mathrm{d}k = 1$.

To gain further insight into the white light speckle angular autocorrelation function it is useful to consider a particular example in which the spatial intensity distribution $S(\boldsymbol{x})$ of the illuminating light is Gaussian with r.m.s. radius $r/2$. It follows from eqs. (42) to (44) that (PEDERSEN [1975b]),

$$C(\Delta \boldsymbol{q}) = \exp\{-r^2 |\Delta \boldsymbol{q}_x|^2 - \sigma_h^2 \Delta q_z^2\}.$$

We further assume that the normalised spectral density $S(k)$ is also Gaussian with an r.m.s. spectral bandwidth W around a mean wavenumber k_0,

$$S(k) = \frac{1}{W\sqrt{2\pi}} \cdot \exp\{-(k - k_0)^2 / 2W^2\}.$$

We now introduce a path vector $\boldsymbol{s} = r\boldsymbol{m}_x + \sigma_h m_z \boldsymbol{e}_z$, where

$$s = |\boldsymbol{s}| = \sqrt{r^2 |\boldsymbol{m}_x|^2 + \sigma_h^2 m_z^2} \tag{47}$$

is the r.m.s. path deviation of the light contributing at the point $\boldsymbol{m}$ in the speckle pattern and $\boldsymbol{e}_z$ is a unit vector in the z direction. Equation (46) for the angular autocorrelation of the intensity can now be evaluated to give

$$C_{\mathrm{w}}(\boldsymbol{s}_1, \boldsymbol{s}_2) = \frac{\exp\left\{-k_0^2 \dfrac{|\boldsymbol{s}_2 - \boldsymbol{s}_1|^2 + 4W^2 |\boldsymbol{s}_1 \times \boldsymbol{s}_2|^2}{1 + 2W^2(s_1^2 + s_2^2) + 4W^4 |\boldsymbol{s}_1 \times \boldsymbol{s}_2|^2}\right\}}{\sqrt{1 + 2W^2(s_1^2 + s_2^2) + 4W^4 |\boldsymbol{s}_1 \times \boldsymbol{s}_2|^2}}. \tag{48}$$

Let the mean path vector be $\boldsymbol{s} = (\boldsymbol{s}_1 + \boldsymbol{s}_2)/2$ and a difference path vector be $\boldsymbol{d} = \boldsymbol{s}_2 - \boldsymbol{s}_1$. In the limit $\boldsymbol{d} = 0$ eq. (48) reduces to

$$C_{\mathrm{w}}(0; \boldsymbol{s}) = \sigma^2/\langle I\rangle^2 = 1/\sqrt{1+(2Ws)^2}, \tag{49}$$

where s is in general given by eq. (47); for normally incident illumination and viewing at the centre of the Fraunhofer plane $s = 2\sigma_h$ and we obtain

$$\sigma^2/\langle I\rangle^2 = 1/\sqrt{1+(4W\sigma_h)^2}. \tag{50}$$

This simple expression clearly shows how the speckle contrast decreases as either the surface roughness σ_h or the illumination bandwidth W increases.

If we now restrict our problem still further by assuming that we are using relatively narrow band sources such that $W \ll k_0$, then eq. (48) reduces to

$$C_{\mathrm{w}}(\boldsymbol{d}; \boldsymbol{s}) \simeq C_{\mathrm{w}}(0; \boldsymbol{s}) \exp\left\{-k_0^2 \frac{d^2 + 4W^2|\boldsymbol{s}\times \boldsymbol{d}|^2}{1+4W^2 s^2}\right\}. \tag{51}$$

In order to show how the speckles are elongated and in what way this elongation depends on the surface and illumination properties, it is convenient to assume that the illumination is normal to the scattering surface and consider paraxial diffraction angles. We then have (Fig. 14),

$$\boldsymbol{s} = \{s_x, s_y, s_z\} \simeq \{r\theta, 0, 2\sigma_h\},$$

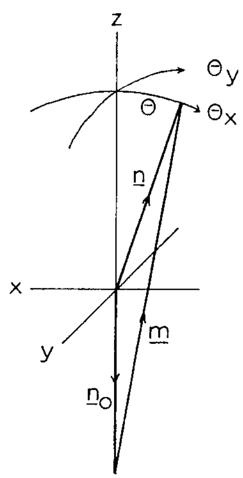

Fig. 14. Diffraction geometry for the calculation of polychromatic speckle correlation with normal incidence (Pedersen [1975b]).

where the x-axis lies along $\boldsymbol{s}_x$ and θ is the polar diffraction angle. Similarly,

$$\boldsymbol{d} \simeq \{r\Delta\theta_x, r\Delta\theta_y, 0\},$$

where $\Delta\theta_x$ and $\Delta\theta_y$ are the components of the change in diffraction angles in the radial x-direction and the azimuthal y-direction respectively. Equation (51) becomes

$$C_{\mathrm{w}}(\Delta\boldsymbol{\theta}; \theta) = C_{\mathrm{w}}(0; \theta) \exp\left\{-k_0^2 s_x^2 \left[\frac{1+4W^2 s_z^2}{1+4W^2 s^2}\left(\frac{\Delta\theta_x}{\theta}\right)^2 + \left(\frac{\Delta\theta_y}{\theta}\right)^2\right]\right\}. \tag{52}$$

It can be seen from eq. (52) that the degree of angular correlation is locally stationary with an elliptical Gaussian correlation function. In the azimuthal direction, the width of the correlation function (i.e. the speckle width) is determined only by the normal diffraction value,

$$\left(\frac{\sigma_\theta}{\theta}\right)_y^2 = \left(\frac{\sigma_\theta}{\theta}\right)_{\mathrm{diffr}}^2 \simeq \frac{1}{2k_0^2 r^2 \theta^2}.$$

In the radial x-direction, the correlation function is lengthened by the process of angular dispersion and we have

$$\left(\frac{\sigma_\theta}{\theta}\right)_x^2 = \left(\frac{\sigma_\theta}{\theta}\right)_{\mathrm{diffr}}^2 + \left(\frac{\sigma_\theta}{\theta}\right)_{\mathrm{disp}}^2.$$

This is illustrated in Fig. 15. The elongation of the speckles in the radial direction due to angular dispersion is given by

$$(\sigma_\theta/\theta)_{\mathrm{disp}}^2 = 2(w_c/k_0)^2, \tag{53}$$

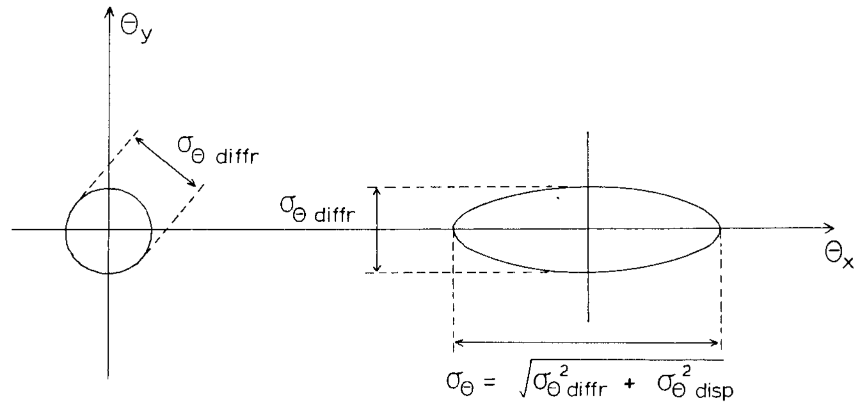

Fig. 15. Illustration of the correlation regions in polychromatic speckle patterns. On axis the correlation region is diffraction-limited and circular. Off axis the angular dispersion causes the region to be radially elongated (Pedersen [1975b]).

where w_c is a correlation bandwidth given by

$$w_c = W/\sqrt{1+(4W\sigma_h)^2}. \quad (54)$$

It is clear from eqs. (53) and (54) that the speckle length in the radial direction depends on both the surface roughness σ_h and the illumination bandwidth W.

The probability density function of intensity in a polychromatic speckle pattern can be found by considering the pattern to be the sum of a number of partially correlated monochromatic speckle patterns, the degree of correlation depending upon position and surface roughness. Using an analysis similar to that given in § 2.3 for the first order statistics of the measured intensity in a normal speckle pattern, it can be shown that the probability density function of intensity in a polychromatic pattern is approximately given by a gamma variate (eq. (30)) or by exact expressions similar to eqs. (27) and (28) (PARRY [1975a]).

The above results indicate that the observation of white light speckle patterns formed in the Fraunhofer plane may produce useful measures of surface roughness and this has been suggested by SPRAGUE [1972] and TRIBILLON [1974], both of whom present preliminary experimental results. However, it must be borne in mind that the above analysis contained a large number of assumptions. In particular it was assumed that single scattering occurred (first Born approximation) and that the surface could be characterised either by a Gaussian distribution of height with non-zero correlation length, or by a "white-noise" distribution; it is not clear whether these assumptions are valid in many practical scattering problems.

Speckle patterns formed in the image plane of a diffuser illuminated by spatially coherent, polychromatic light have been studied by ELBAUM, GREENBAUM and KING [1972], GEORGE and JAIN [1972, 1973, 1974], and MCKECHNIE [1975]. The main application here is the reduction of speckle in the images of diffuse objects. The analysis of the statistics is very similar to the above case for the Fraunhofer plane. The main results are (i) the speckle pattern is statistically stationary for a uniform diffuse object provided that the aberrations of the imaging system are not field-dependent, (ii) the speckle contrast decreases as the bandwidth W of the illumination increases and as the surface roughness σ_h increases, (iii) the speckle pattern depends on the aberrations of the imaging system. GEORGE and JAIN [1974] and MCKECHNIE [1975] have given detailed analyses of the statistics using Goodman's model of the scattering surface.

§ 4. Surface-dependent Features of Speckle Patterns

In earlier sections we have stressed that speckle patterns resulting from certain surfaces being illuminated by monochromatic spatially coherent light are essentially independent of detailed surface properties. The main requirements were (i) the surface does not alter the polarisation of the incident light, (ii) a large number of scattering centres contribute to the amplitude at any point in the observation plane, and (iii) the phase of the scattered wave is random in the interval $-\pi$ to π (or $\sigma_h > \lambda$). In monochromatic light it was shown that the value of the surface roughness σ_h did not influence the speckle statistics provided that condition (iii) was followed, although in polychromatic light σ_h does influence the speckle statistics.

In this section we shall consider the statistics of speckle patterns formed when the above requirements are not fulfilled and we shall see that *in general* the properties of speckle patterns depend in a very complicated way on surface properties.

4.1. DEPOLARISING SURFACES

When plane-polarised monochromatic light is incident on a scattering medium, the transmitted or reflected amplitude in general consists of components of the field that are parallel and perpendicular to the incident field; these components have in general unequal intensities and an arbitary correlation factor. A discussion of mechanism of depolarisation and its relation to the scattering medium is outside the scope of this article (BECKMANN and SPIZZICHINO [1963]).

As far as the effect of depolarisation on the statistics of the resultant speckle pattern is concerned, we may reduce the problem to that of an addition of two partially correlated speckle patterns of unequal intensity (BARAKAT [1973], GOODMAN [1975a, c]). Let A_1, I_1 and A_2, I_2 be the amplitude and intensity of the speckle patterns produced by the parallel and perpendicular components respectively. The correlation coefficient for the amplitude is given by

$$\mu_{mn} = \frac{\langle A_m A_n \rangle}{\sqrt{\langle |A_m|^2\rangle\langle |A_n|^2\rangle}} \quad (m, n = 1, 2).$$

Experimentally we can measure a correlation matrix of intensities (CHAKRA-

BORTY [1973], GEORGE, JAIN and MELVILLE [1975])

$$c_{mn} = \frac{\langle I_m I_n \rangle - \langle I_m \rangle \langle I_n \rangle}{\langle I_n \rangle \langle I_m \rangle}.$$

However, for a large number of scattering centres the complex amplitude in each speckle pattern is a complex Gaussian process and using the results of REED [1962]

$$c_{mn} = |\mu_{mn}|^2,$$

or

$$\mu_{mn} = \sqrt{c_{mn}} \cdot \exp(\mathrm{i}\phi_{mn}),$$

where ϕ_{mn} is a phase factor. It turns out that the phase factor is of no consequence in this case, but has an important influence when a similar analysis is applied to the addition of three or more correlated speckle patterns GOODMAN [1975a, c]). The hermitian coherency matrix whose elements are $\langle A_m A_n^* \rangle$ can therefore be written

$$J = \begin{bmatrix} \langle I_1 \rangle & \sqrt{\langle I_1 \rangle \langle I_2 \rangle}\mu_{12} \\ \sqrt{\langle I_1 \rangle \langle I_2 \rangle}\mu_{12}^* & \langle I_2 \rangle \end{bmatrix}.$$

In problems where we have the sum of correlated random processes, the standard method of solution involves expanding the random process in an orthogonal series such that the sum is of *independent* random processes; standard theorems of probability theory can then be applied to yield $p(I)$, the probability density function. Applying this to our example yields (GOODMAN [1975a, c]), for distinct eigenvalues,

$$p(I) = \frac{\exp(-I/\varepsilon_1)}{\varepsilon_1 - \varepsilon_2} - \frac{\exp(-I/\varepsilon_2)}{\varepsilon_1 - \varepsilon_2}, \qquad I \geqq 0,$$
$$= 0 \qquad \text{otherwise,}$$

or for identical eigenvalues, $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$

$$p(I) = (I/\varepsilon_0^2)\exp(-I/\varepsilon_0), \qquad I \geqq 0,$$
$$= 0 \qquad \text{otherwise,}$$

where ε_n are the eigenvalues of the coherency matrix. An example is given in Fig. 16 for the cases of $c_{12} = 0$ ($\varepsilon_1 = \varepsilon_2 = 0.5$), $c_{12} = 0.6$ ($\varepsilon_1 = 0.887$, $\varepsilon_2 = 0.113$) and $c_{12} = 1$ ($\varepsilon_1 = 1$, $\varepsilon_2 = 0$) with the assumption that the two mean intensities are equal.

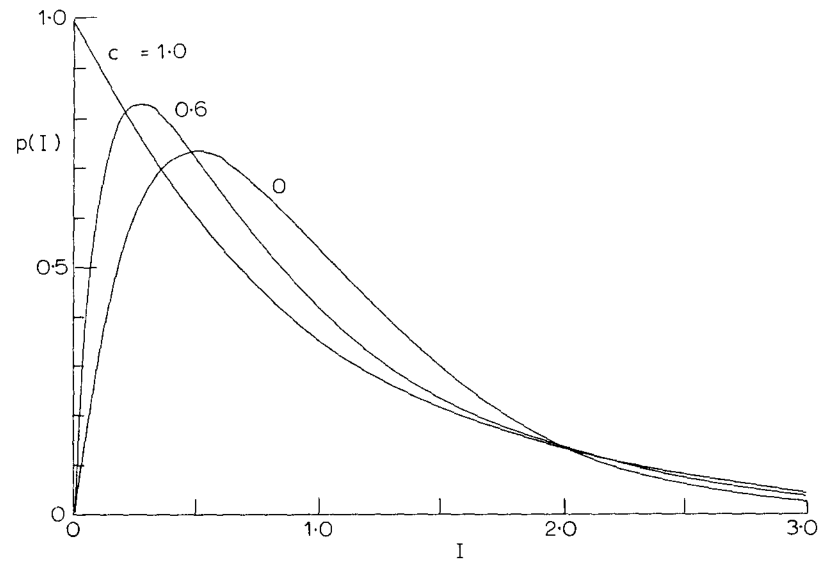

Fig. 16. The probability density functions for the intensity of the sum of two partially correlated normal speckle patterns with equal mean intensity and $c_{12} = 0$, 0.6 and 1.0 (GOODMAN [1975a, c]).

4.2. A SMALL NUMBER OF SCATTERERS

If only a small number of scatterers contribute to the amplitude at a point in the observation plane then the central limit theorem cannot be applied and the complex amplitude will not have a complex Gaussian distribution. In this case the statistics of the scattered field will depend on the statistics of the scattering medium. This dependence is discussed below. A small number of scatterers also implies that the autocorrelation function of the amplitude of the scattered wave immediately behind the scatterer is not small compared to the dimensions of the illuminated area. In all cases considered so far in this article we have assumed that this correlation function is small in extent relative to the illuminated area (i.e. diffusers relatively of fine structure) and so we first consider the effect of allowing this autocorrelation function to increase in area, but at the same time assuming that the number of correlation areas is sufficiently large to apply the central limit theorem.

Consider the optical system in Fig. 17. A plane wave is incident on a random medium which imposes a complex amplitude distribution $A(\xi)$ at the entrance pupil of a lens whose pupil function is $H(\xi)$. If the lens is aberration-free then the amplitude in the observation (Fraunhofer) plane

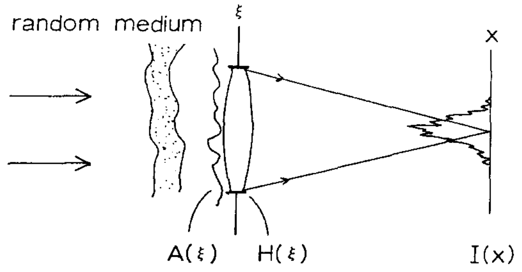

Fig. 17. Formation of a speckle pattern in the Fraunhofer plane for a random complex amplitude with a correlation function of non-zero extent.

is the *finite* Fourier transform of $A(\xi)$, but in general the amplitude in the observation plane is the Fourier transform of the product $A(\xi)\ H(\xi)$. We have included this dependence on the pupil function as this problem first arose in an application in astronomy where the atmosphere is the random medium and the effect of telescope aberrations is of practical interest (DAINTY [1973, 1974]). The autocorrelation function of the amplitude $A(\xi)$ is defined as

$$C_A(\xi) = \langle A^*(\xi_1)A(\xi_1+\xi)\rangle$$

and is independent of ξ_1 if the random wavefront $A(\xi)$ is statistically stationary. Because the extent of $C_A(\xi)$ is not considered to be small compared with the limiting aperture of the pupil function, the intensity distribution in the observation plane (the Fraunhofer plane of $A(\xi)$ and also the image plane of the point source) is concentrated in a region near to the optic axis; within this region a speckle-like pattern is seen.

The Wiener spectrum of the intensity distribution is equal to $\langle|i(u)|^2\rangle$, where $i(u)$ is the Fourier transform of the intensity distribution $I(x)$. Thus the Wiener spectrum is given by

$$W(u) = \int_{-\infty}^{\infty}\int H^*(\xi_1)H(\xi_2)H(\xi_1+\xi)H^*(\xi_2+\xi) \times \langle A^*(\xi_1)A(\xi_2)A(\xi_1+\xi)A^*(\xi_2+\xi)\rangle\, d\xi_1\, d\xi_2, \tag{55}$$

where $u = \xi/\lambda f$. It is clear from this equation that the Wiener spectrum of the speckle depends on the fourth order moment of the complex amplitude of the scattered field. If this complex amplitude is a "white" noise process, the fourth order moment can be expressed in terms of delta functions, and the expression for the Wiener spectrum reduces to that given by either

eq. (15) or (21) in §2. To consider the effect of the autocorrelation of the complex amplitude on the Wiener spectrum of the speckle intensity, we consider a particular example in which it is assumed that $A(\xi)$ is a complex Gaussian process (DAINTY [1973]). The fourth order moment can then be written in terms of the autocorrelation function and eq. (55) becomes, after some rearrangement,

$$W(u) = |C_A(\xi)|^2 \left| \int_{-\infty}^{\infty} H^*(\xi_1)H(\xi_1+\xi)\, d\xi_1 \right|^2 + \int_{-\infty}^{\infty}\int |C_A(\xi_2-\xi_1)|^2 H^*(\xi_1)H(\xi_2)H(\xi_1+\xi)H^*(\xi_2+\xi)\, d\xi_1\, d\xi_2. \tag{56}$$

This expression is essentially the same as one given by ENLOE [1967] for an analogous case where a partially resolved diffuser is imaged by an optical system and also as eq. (59) below. The first term on the right-hand side of eq. (56) is governed by the average intensity in the observation plane and the second term describes the intensity fluctuation. For a "white" noise field $C_A(\xi) = \delta(\xi)$, and the intensity fluctuation term becomes identical to that given in eqs. (15) and (21) in §2.

In Fig. 18 we give an example of the effect of $C_A(\xi)$ on the Wiener spectrum of the intensity fluctuation (second term in eq. (56)). The autocorrelation function is assumed (for computational convenience) to have a top-hat form with a diameter that is some fraction R of the diameter of the optical system (DAINTY [1974]). In Fig. 18(a) and (b) the effect of defocus is shown for $R = 0.2$ and $R = 0.1$ respectively. Clearly the speckle pattern is independent of lens aberration only in the limit $R \to 0$ (i.e. a "white" noise field $A(\xi)$).

The problem of finding the first order statistics when N is finite was first examined by Lord RAYLEIGH [1919] and more recently his results were applied by BURCH [1969]. More general analyses with application to the scattering by liquid crystals have been given by JAKEMAN and PUSEY [1973a, b, 1975] and a summary of their results is presented below.

It is assumed that the scatterer is a deep phase screen such that the phase of the scattered light immediately after the screen has a Gaussian distribution with a variance $\sigma_\beta^2 \gg 1$. The complex amplitude in the Fraunhofer plane is given by

$$A(x, y) = \sum_{j=1}^{N} \alpha_j(x, y) \exp(i\beta_j), \tag{57}$$

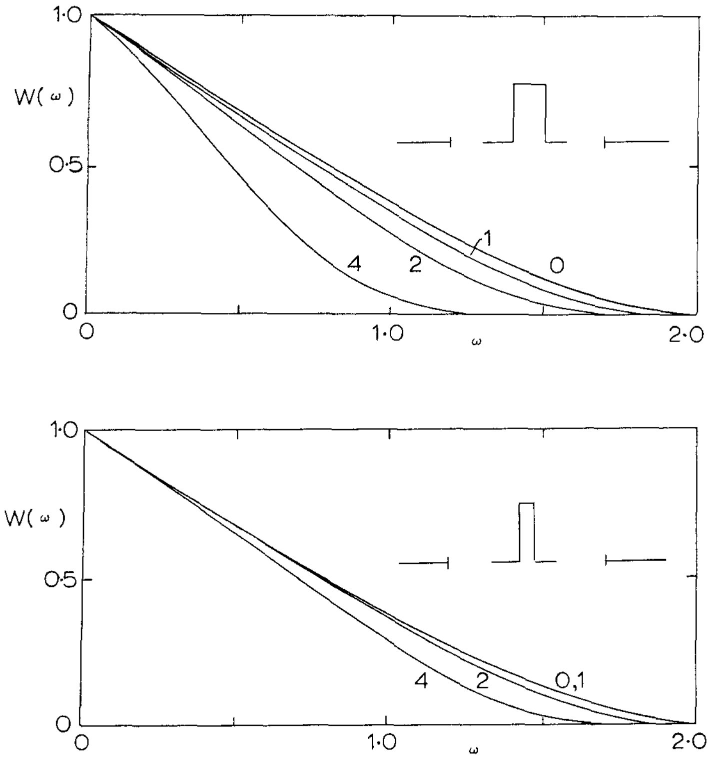

Fig. 18. Wiener spectra of the intensity fluctuation for speckle patterns observed in the Fraunhofer plane of a defocussed lens for defocus values of 0, $1/\pi$, $2/\pi$ and $4/\pi$ wavelengths. Upper (a), $R = 0.2$; lower (b), $R = 0.1$ The parameter R is equal to the diameter of the correlation area relative to that of the optical system (DAINTY [1974]).

where N is *finite*, $\alpha_j(x, y)$ is a scattering factor, and β_j is the random phase from the jth scatterer and is independent of the phases of all other scatterers. Equation (57) describes a finite random walk with variable step length studied by Lord RAYLEIGH [1919] and others. Assuming that all the α_j can be described by the same distribution function, the following expressions for the first two moments of the intensity can be found:

$$\langle I\rangle = N\langle\alpha^2\rangle,$$
$$\langle I^2\rangle = N\langle\alpha^4\rangle + 2N(N-1)\langle\alpha^2\rangle^2.$$

Higher moments can also be found. Evaluating the moments of α_j yields an expression for the ratio of the variance of the speckle intensity to the

square of the mean intensity (JAKEMAN and PUSEY [1975]),

$$\frac{\sigma^2}{\langle I\rangle^2} = 1 - \frac{2}{N} + \frac{\sigma_\beta{}^2}{4N}\exp\left\{\frac{k^2 r_\beta^2 \sin^2\theta}{\sigma_\beta^2}\right\}, \tag{58}$$

where $k = 2\pi/\lambda$, r_β is the phase correlation length, and θ is the angle observation (for normal incidence). Clearly as $N \to \infty$, $\sigma^2/\langle I\rangle^2$ tends to the Gaussian value of unity. However for finite N this ratio (the speckle "contrast") depends on the angle of observation, the variance σ_β^2 and correlation scale length r_β of the phase immediately after the scatterer and of course on N itself. Some experimental results (PUSEY and JAKEMAN [1975]) for the variation of speckle contrast with angle of observation θ and the number of scatterers for a liquid crystal scatterer are given in Figs. 19 and 20 respectively; these broadly confirm eq. (58).

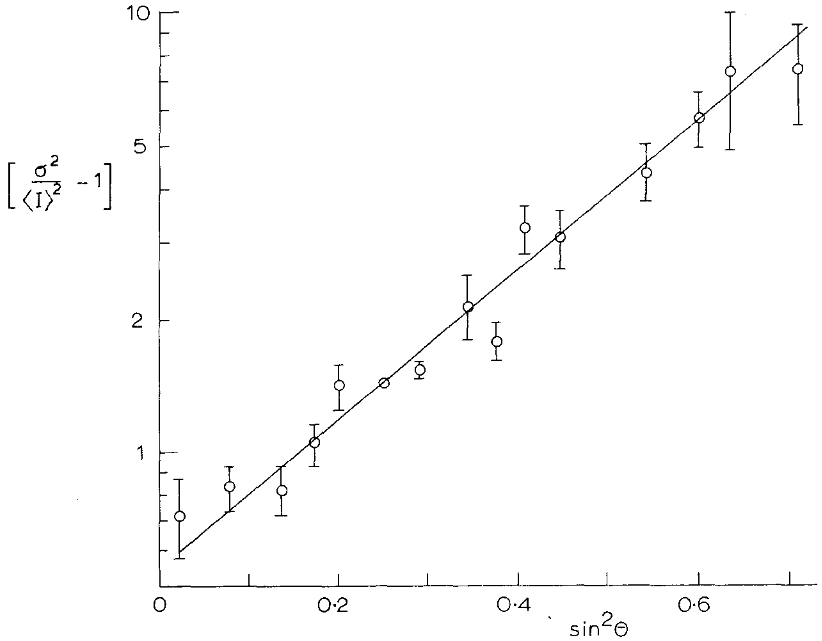

Fig. 19. Angular dependence of the speckle contrast for a small number of scatterers (PUSEY and JAKEMAN [1975]).

The normalised angular autocorrelation function of the intensity,

$$C(\theta, \theta') = \frac{\langle I(\theta)I(\theta')\rangle}{\langle I(\theta)\rangle\langle I(\theta')\rangle}$$

can also be found using eq. (57) and evaluating the appropriate moments

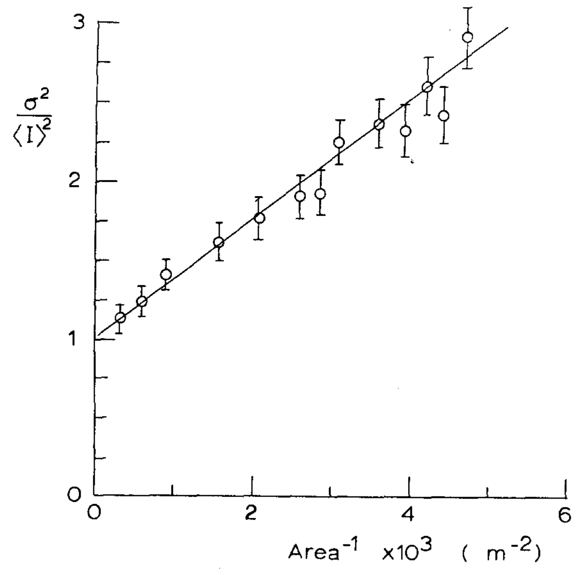

Fig. 20. Dependence of speckle contrast on illuminated area of diffuser for a fixed size of inhomogeneity (PUSEY and JAKEMAN [1975]).

of α_j yields (JAKEMAN and PUSEY [1975])

$$C(\theta, \theta') = \left(1 - \frac{1}{N}\right) C_{\infty}(\Delta\theta) + \frac{\sigma_\beta^2}{4N} \exp\{-k^2 r_\beta^2 \Delta\theta^2/16\}$$

$$\times \exp\left\{\frac{k^2 r_\beta^2}{16\sigma_\beta^2}[(\theta+\theta')^2 + 2\Delta\theta^2]\right\}, \qquad (59)$$

where small θ and θ' have been assumed for simplicity, $\Delta\theta = \theta - \theta'$, and where $C_\infty(\Delta\theta)$ is the autocorrelation function of the normal speckle pattern produced when $N \rightarrow \infty$ as given by eq. (14). Clearly eq. (59) gives the correct result of $C(\theta, \theta') \rightarrow C_\infty(\Delta\theta)$ as $N \rightarrow \infty$. However for finite N two characteristic scale lengths appear in the autocorrelation function. The first is the normal speckle "size" governed by the total extent of the scattering volume, whilst the second depends on the variance and scale length of the phase immediately behind the scatterer. This result is essentially the same as that given in eq. (56) but derived and used in a different context (DAINTY [1973]). Some experimental measurements of $C(\theta, \theta')$ in the scattering by liquid crystals are shown in Fig. 21 and broadly confirm the above theoretical results. The statistics of speckle patterns produced when only a *finite* number of scatterers contribute to the pattern at any point clearly depend

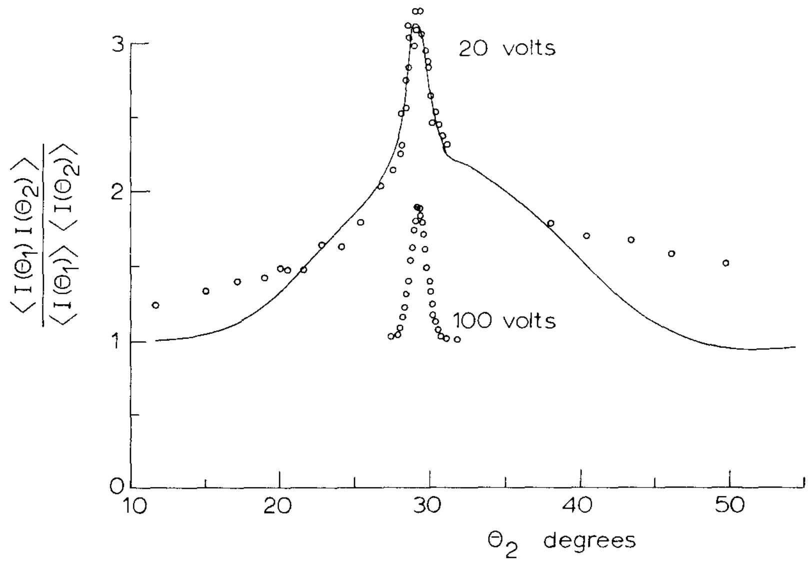

Fig. 21. Angular autocorrelation functions in the Fraunhofer plane for speckle patterns produced by many scatterers (100 V) and a few scatterers (20 V) (PUSEY and JAKEMAN [1975]).

on the properties of the scattering surface and may be of use in determining these properties.

4.3. SLIGHTLY ROUGH SURFACES

The statistics of speckle patterns considered throughout this article have been derived on the assumption that the surface roughness is greater than one wavelength. If $\sigma_h < \lambda$ then the speckle pattern will contain information on the surface properties; several authors have suggested that useful surface parameter information is contained in speckle patterns produced by slightly rough surfaces (ALLEN and JONES [1963], CRANE [1970], GOODMAN [1973, 1975b, c], PEDERSEN [1974], FUJII and ASAKURA [1974b, c], WELFORD [1975]).

It is convenient to first examine the speckle patterns formed in coherent light in the image plane of a slightly rough diffuser. Such a speckle pattern can be considered to consist of a uniform beam arising from a specularly reflected component added coherently to a diffusely reflected beam which forms a normal speckle pattern. The relative intensities of the two components will depend on the properties of the scatterer. The probability density function of the intensity of a speckle pattern produced in the image plane of a slightly rough diffuser will be similar to that obtained when a

uniform beam is added coherently to a speckle pattern. This problem has already been investigated by GOODMAN [1967] and DAINTY [1972]. The real and imaginary parts of the complex amplitude in the speckle pattern have a Gaussian distribution with *non-zero* mean and the distribution of intensity and phase can be found using the appropriate probability transformation (DAINTY [1972]).

In Fig. 22(a) we show some experimental measurements of the probability distribution of intensity $p(I)$ for four surfaces, and in Fig. 22(b) theoretical curves for $p(I)$ are shown for a range of specular/diffuse beam ratios. The agreement between the overall forms of the two sets of curves is good and indicates that the simple approach outlined above may be

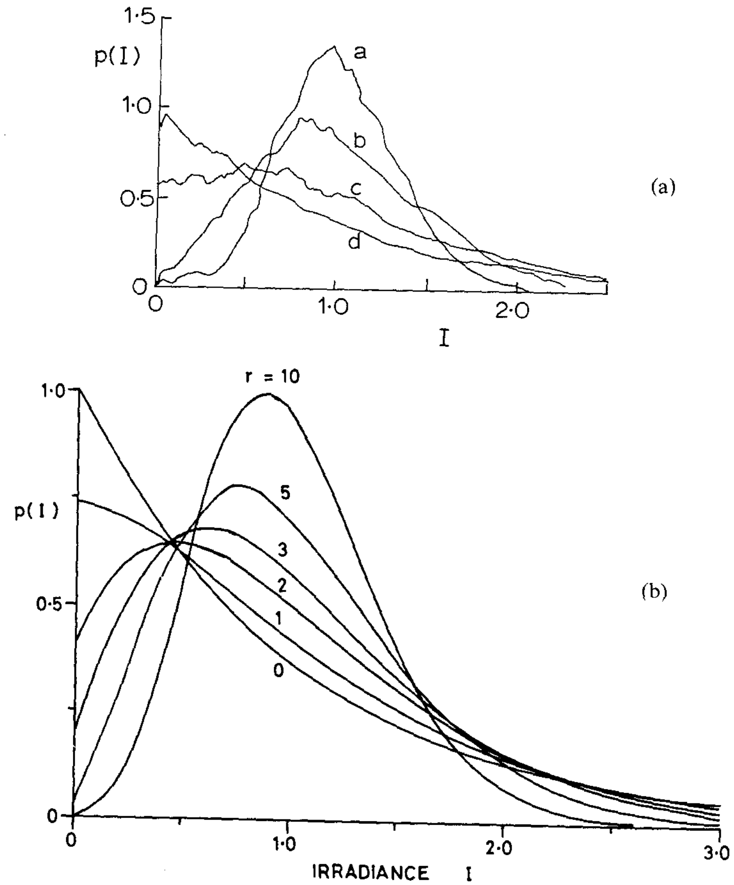

Fig. 22. (a) Experimental probability distributions of speckle intensity in the images of slightly rough diffusers (FUJII and ASAKURA [1974b]). (b) Probability density functions for a range of specular/diffuse beam ratios (DAINTY [1972]).

adequate for practical purposes. The main difficulty lies in establishing a link between the surface parameters and the ratio of specular/diffuse transmittance (or reflectance). In order to establish this link it is again necessary to use a surface model that gives a Gaussian distribution of phase in the scattered wave immediately after the diffuser; the standard deviation of the phase is σ_ϕ and the scale length of the phase correlation function is r_ϕ. Using this model FUJII and ASAKURA [1974c] evaluated $\sigma/\langle I\rangle$ as a function of both the relevant surface parameters and the degree of spatial coherence.

Their theoretical and experimental results are summarised in Fig. 23. The surface parameter of importance is (for reflected light)

$$\frac{\sigma_\phi^2}{r_\phi} = \left(\frac{2\pi}{\lambda}\right)^2 \frac{\sigma_h^2}{r_\phi} = \left(\frac{2\pi}{\lambda}\right)^2 \sigma_h \left(\frac{\sigma_h}{r_\phi}\right). \tag{60}$$

The final term on the right hand side of eq. (60), (σ_h/r_ϕ) is closely related to the r.m.s. gradient of the surface height distribution and this equation indicates that a simple measure of speckle contrast in the image cannot uniquely determine the surface roughness.

Recently OHTSUBO and ASAKURA [1975] have observed that the speckle contrast is somewhat smaller in the image plane than in out-of-focus planes. GOODMAN [1975b, c] has shown that this is because the scattered field does not have circular Gaussian statistics (as assumed above) and that

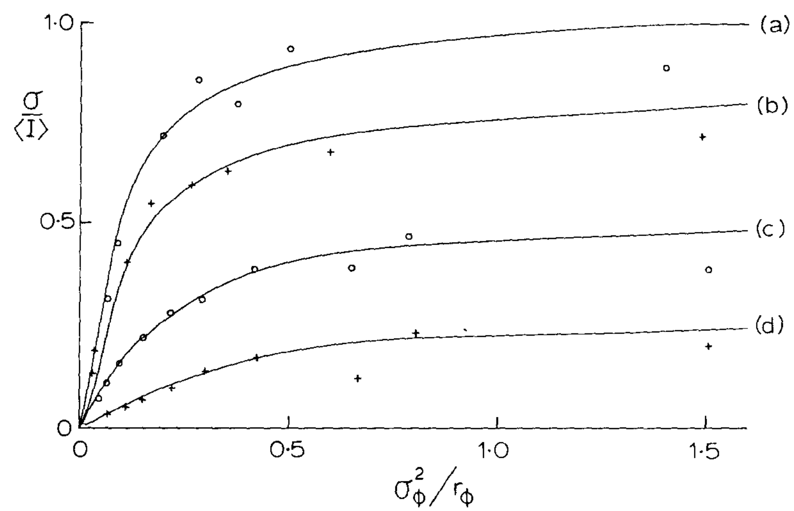

Fig. 23. Speckle contrast as a function of roughness characteristic σ_ϕ^2/r_ϕ for four conditions of partial spatial coherence (a) – (d) (FUJII and ASAKURA [1974c]).

the degree of non-circularity depends on the surface roughness. The dip in speckle contrast through focus is due to change from circular to non-circular statistics and back again.

It may also be useful to study the intensity at the centre of the Fraunhofer plane within the first bright ring of the Airy disc for a slightly rough surface. A similar statistical variation to that shown in Fig. 22 would be obtained, but in this case one would not expect on simple physical grounds any strong dependence on the r.m.s. gradient of the surface height. Further advances are awaited on this aspect of scattering from rough surfaces.

§ 5. Concluding Remarks

We began this review article by stressing that in a large number of practical situations *normal* speckle patterns are produced; the statistics of these speckle patterns are well-established both theoretically and experimentally and in particular they do not depend on the detailed scattering properties of the scattering medium. When examining the effects of polychromatic illumination on speckle statistics we found the value of the r.m.s. height variation of a scattering surface strongly influenced the statistical properties of the scattered intensity. Finally in §4 it was seen that the general relationship between the scattered intensity and the scattering surface may be very complicated, even for surfaces with a Gaussian distribution of surface heights. However, it must be stressed that through the whole review we have glossed over the *details* of the interaction of an electromagnetic wave with a scattering medium and have used relatively naïve models of scattering surfaces. Hopefully this unsatisfactory state of affairs in the subject as a whole will be rectified by the advances of future workers.

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II

HIGH-RESOLUTION TECHNIQUES IN OPTICAL ASTRONOMY

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