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OBJECT RECONSTRUCTION FROM PHOTON-LIMITED CENTROIDED DATA OF RANDOMLY TRANSLATING IMAGES

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ABSTRACT

Centroiding is investigated as a simple and computationally fast technique of image reconstruction, at low light level, of a randomly translating image. The detected frames are sorted by their number of photons, centroided, separated averages performed and then compared with the usual way of centroiding frames. An algorithm for retrieving the phase for one-dimensional centroided imaging is presented and computer simulated data is used to test the theory and the reconstruction technique.

1 - INTRODUCTION

The diffraction-limited image resolution of a ground-based 4 metre telescope operating at 400 nm is approximately 0.025" (arc second). Atmospheric turbulence, however, makes the image of an unresolved star broader ~ 1.0" (one second of arc) and exhibit a granular structure resembling a speckle pattern.

A number of techniques have been developed to retrieve the diffraction-limited image by recording short-exposure frames of these speckle patterns. The SHIFT-AND-ADD (SAA) method[1-5] is one of the proposed techniques. It relies on the proposition that each frame consists of many distorted replicas of the true image and that an improved estimate of the image could be found by superimposing these distorted replicas. The superposition is carried out by considering that the brightest part of a speckle image is, most likely, a distorted version of the brightest part of the true image. In each frame, the brighter speckles are found and by shifting and superimposing them at the centre of the frame (SHIFT-AND-ADD) the imperfections of the individual images tend to average out.

At very low light levels, for example, the mean number of photons/frame $\overline{N} < 10$ photons/frame, however, each of these distorted images may have only a few photons and no "bright" speckle can be chosen in order to implement the SAA technique. As a first approach on how the shift-and-add (SAA) technique behaves, at very low light levels, a theoretical study is presented focusing on centroiding photon-limited data emitted from a **randomly translating image** . A relationship between these centroided images and the stationary normalized image $I(\mathbf{r})$ as well as a phase reconstruction algorithm is presented for the case of one-dimensional object. Computer simulated photon data emitted from a binary star system is used to assess the theory and the reconstruction algorithm.

2 - THEORY OF CENTROIDING DATA OF RANDOMLY TRANSLATING IMAGES

To "freeze" a randomly moving image, photons, that are all supposed to be emanated from the same randomly translating image, are detected during a series of short time intervals(frames). To retrieve the stationary image one should, for each frame, shift the photon vectors by the amount that **the true centroid of the image** is displaced in respect to the centre of the frame and average(add) over all the frames. The true centroid and hence the true shift vector $\mathbf{c}_k$, however, is unknown and another shift vector $\mathbf{R}_k$ must be determined (see Fig. 1). As an estimator for the true centroid, the centroid vector of the detected photons can be evaluated, all the photon vectors are then shifted by this estimator and an average image of many such frames is formed.

The relationship between this estimated image and the normalized stationary image $I(\mathbf{r})$ can be derived assuming that the N-photon data $d_k(\mathbf{x})$ detected on the k^{th} frame is modeled[6] as an inhomogeneous Poisson process mathematically described as:

$$d_k(\mathbf{x}) = \sum_{j=1}^{N} \delta(\mathbf{x} - \mathbf{x}_j) \tag{1}$$

Let $\widetilde{D}_k(\mathbf{u}, \mathbf{x}_1, ..., \mathbf{x}_N)$ be the Fourier transform of $d_k(\mathbf{x})$

$$\begin{aligned} \widetilde{D}_k(\mathbf{u}, \mathbf{x}_1, ..., \mathbf{x}_N) &= \sum_{j=1}^{N} \int_{-\infty}^{+\infty} \delta(\mathbf{x} - \mathbf{x}_j)\, exp(-i2\pi \mathbf{u} \cdot \mathbf{x}) d\mathbf{x} \\ &= \sum_{j=1}^{N} exp(-i2\pi \mathbf{u} \cdot \mathbf{x}_j) \end{aligned} \tag{2}$$

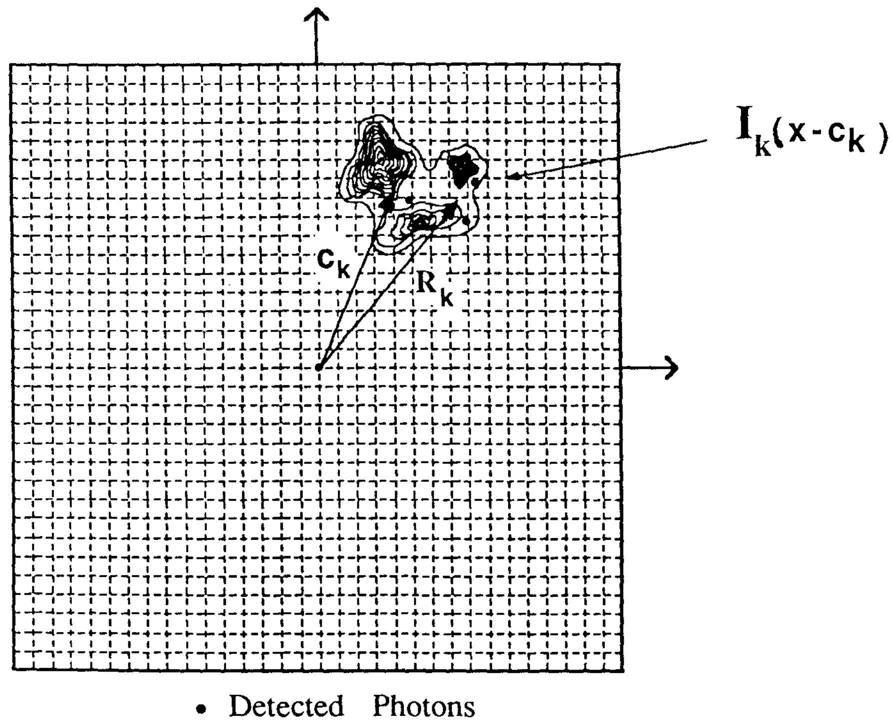

Figure 1 - Typical frame showing the true and the estimated shift vectors - $\mathbf{c}_k$ and $\mathbf{R}_k$

The N photon centroid vector $\mathbf{R}_k$ of the k^{th} frame (Figure 1) is defined as

$$\mathbf{R}_k = \frac{1}{N}\sum_{j=1}^{N}\mathbf{x}_j \tag{3}$$

Substituting in terms of the centroid coordinates $\mathbf{r}_j = \mathbf{x}_j - \mathbf{R}_k$ (Fig. 1) we have

$$\begin{aligned}\widetilde{D}_k(\mathbf{u},\mathbf{x}_1,...,\mathbf{x}_N) &= exp(-i2\pi\mathbf{u}\cdot\mathbf{R}_k)\sum_{j=1}^{N} exp(-i2\pi\mathbf{u}\cdot\mathbf{r}_j) \\ &= exp(-i2\pi\mathbf{u}\cdot\mathbf{R}_k)\,\widetilde{D}_k^c(\mathbf{u},\mathbf{r}_1,...\mathbf{r}_N)\end{aligned} \tag{4}$$

where $\widetilde{D}_k^c(\mathbf{u},\mathbf{r}_1,...\mathbf{r}_N) = \widetilde{D}_k^c(\mathbf{u},\mathbf{x}_1,...\mathbf{x}_N)$ is the Fourier transform of the centroided $\widetilde{D}_k(\mathbf{u},\mathbf{x}_1,...\mathbf{x}_N)$.

The relation between the centroided and non-centroided data spectrum is therefore:

$$\begin{aligned}\widetilde{D}_k^c(\mathbf{u},\mathbf{x}_1,...,\mathbf{x}_N) &= exp(+i2\pi\mathbf{u}\cdot\mathbf{R}_k)\,\widetilde{D}_k(\mathbf{u},\mathbf{x}_1,...\mathbf{x}_N) \\ &= exp(+i2\pi\mathbf{u}\cdot\mathbf{R}_k)\sum_{j=1}^{N} exp(-i2\pi\mathbf{u}\cdot\mathbf{x}_j) \\ &= \sum_{j=1}^{N} exp(+i2\pi\mathbf{u}\cdot\frac{\mathbf{x}_1}{N}).......exp(-i2\pi\mathbf{u}\cdot[1-\frac{1}{N}]\mathbf{x}_j).......exp(+i2\pi\mathbf{u}\cdot\frac{\mathbf{x}_N}{N})\end{aligned} \tag{5}$$

Due to the dependence of $\widetilde{D}_k^c(\mathbf{u},\mathbf{x}_1,...,\mathbf{x}_N)$ on the photon coordinates $\mathbf{x}_j$ the ensemble average of these $\widetilde{D}_k^c(\mathbf{u},\mathbf{x}_1,...,\mathbf{x}_N)$ can be performed taking into account that the $\mathbf{x}_j's$ are independent random variables and that consequently the joint probability density, $p(\mathbf{x}_1,...,\mathbf{x}_N)$, of detecting N photons between coordinates $\mathbf{x}_1,.....,\mathbf{x}_N$ and $\mathbf{x}_1 + d\mathbf{x}_1,.....,\mathbf{x}_N + d\mathbf{x}_N$ is the product of

the independent probability densities of having one photon detected between $\mathbf{x}_1$ and $\mathbf{x}_1 + d\mathbf{x}_1$ times the probability density of detecting another photon between $\mathbf{x}_2$ and $\mathbf{x}_2 + d\mathbf{x}_2$ and so on. Hence

$$p(\mathbf{x}_1, ..., \mathbf{x}_N) d\mathbf{x}_1 \dots d\mathbf{x}_N = p(\mathbf{x}_1).p(\mathbf{x}_2).....p(\mathbf{x}_N)\, d\mathbf{x}_1 \dots d\mathbf{x}_N \tag{6}$$

where $p(\mathbf{x}_j)$ is equal to the normalized intensity $\mathcal{I}_k(\mathbf{x}_j)$ at coordinate $\mathbf{x}_j$ and frame k i.e.:

$$p(\mathbf{x}_j) = \mathcal{I}_k(\mathbf{x}_j) \tag{7}$$

Let $\mathbf{c}_k$ represent the true centroid shift vector at frame k. The function $\mathcal{I}_k(\mathbf{x}_j)$ at frame k and coordinate $\mathbf{x}_j$ is related to the stationary normalized image $I(\mathbf{x}_j)$ (Fig. 1) by

$$\mathcal{I}_k(\mathbf{x}_j) = I(\mathbf{x}_j - \mathbf{c}_k) \tag{8}$$

As a consequence

$$\begin{aligned} p(\mathbf{x}_j) &= \mathcal{I}_k(\mathbf{x}_j) \\ &= I(\mathbf{x}_j - \mathbf{c}_k) \\ &= p(\mathbf{x}_j, \mathbf{c}_k) \end{aligned} \tag{9}$$

where $p(\mathbf{x}_j, \mathbf{c}_k)$ is a normalized probability density that depends both on $\mathbf{x}_j$ and on the random true centroid vector $\mathbf{c}_k$.

The average of $\widetilde{D}^c_k(\mathbf{u}, \mathbf{x}_1, ..., \mathbf{x}_N)$ over the ensemble of frames has therefore to be done in two steps: first averaging over the detected photon coordinates $\mathbf{x}_j$ and then over the true centroid vector $\mathbf{c}_k$. Lets represent by $\widetilde{D}^c_k(\mathbf{u}, \mathbf{c}_k)$ the result after averaging over $\mathbf{x}_1, ..., \mathbf{x}_N$:

$$\begin{aligned} \widetilde{D}^c_k(\mathbf{u}, \mathbf{c}_k) &= \left\langle \widetilde{D}^c_k(\mathbf{u}, \mathbf{x}_1, ..., \mathbf{x}_N) \right\rangle_{\mathbf{x}_1, ..., \mathbf{x}_N} \\ &= \int_{-\infty}^{+\infty} p(\mathbf{x}_1, \mathbf{c}_k)....p(\mathbf{x}_N, \mathbf{c}_k)\, \widetilde{D}^c_k(\mathbf{u}, \mathbf{x}_1, ..., \mathbf{x}_N)\, d\mathbf{x}_1....d\mathbf{x}_N \\ &= \int_{-\infty}^{+\infty} I(\mathbf{x}_1 - \mathbf{c}_k).....I(\mathbf{x}_N - \mathbf{c}_k)\, \widetilde{D}^c_k(\mathbf{u}, \mathbf{x}_1, ..., \mathbf{x}_N)\, d\mathbf{x}_1.....d\mathbf{x}_N \end{aligned} \tag{10}$$

Substituting (5) into (10) one get:

$$\begin{aligned} \widetilde{D}^c_k(\mathbf{u}, \mathbf{c}_k) = \sum_{j=1}^{N} \int_{-\infty}^{+\infty} & I(\mathbf{x}_1 - \mathbf{c}_k)\, exp(+i2\pi\mathbf{u} \cdot \frac{\mathbf{x}_1}{N})\, d\mathbf{x}_1........ \\ & ......\int_{-\infty}^{+\infty} I(\mathbf{x}_j - \mathbf{c}_k)\, exp(-i2\pi\mathbf{u} \cdot \mathbf{x}_j[1 - 1/N])\, d\mathbf{x}_j.......... \\ & ......\int_{-\infty}^{+\infty} I(\mathbf{x}_N - \mathbf{c}_k)\, exp(+i2\pi\mathbf{u} \cdot \frac{\mathbf{x}_N}{N})\, d\mathbf{x}_N \end{aligned} \tag{11}$$

The integrals in (11) can be evaluated easily giving:

$$\int_{-\infty}^{+\infty} I(\mathbf{x}_j - \mathbf{c}_k) exp(-i2\pi\vec{\alpha} \cdot \mathbf{x}_j) d\mathbf{x}_j = exp(-i2\pi\vec{\alpha} \cdot \mathbf{c}_k)\, \widetilde{I}(\vec{\alpha}) \tag{12}$$

where $\widetilde{I}(\vec{\alpha})$ is the Fourier transform of $I(\mathbf{r})$ and $\vec{\alpha}$ can either be equal to $-\mathbf{u}/N$ or to $+\mathbf{u}\,[1-1/N]$.

Using result (12) the jth term of the sum in (11) is:

$$= \widetilde{I}(-\mathbf{u}/N)\, exp(+i2\pi\frac{\mathbf{u}}{N}\cdot\mathbf{c}_k)\,......\widetilde{I}(\mathbf{u}[1-1/N])\, exp(-i2\pi\,\frac{N-1}{N}\mathbf{u}\cdot\mathbf{c}_k)\,.....\,\widetilde{I}(-\mathbf{u}/N)\, exp(+i2\pi\frac{\mathbf{u}}{N}\cdot\mathbf{c}_k)$$

$$= \widetilde{I}(\mathbf{u}[1-1/N])\, exp(-i2\pi\,\frac{N-1}{N}\,\mathbf{u}\cdot\mathbf{c}_k)\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1}\ \left[exp(+i2\pi\frac{\mathbf{u}}{N}\cdot\mathbf{c}_k)\right]^{N-1}$$

$$= \widetilde{I}(\mathbf{u}[1-1/N])\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1}\quad exp(+i2\pi\left[\frac{N-1}{N}-\frac{N-1}{N}\right]\mathbf{u}\cdot\mathbf{c}_k)$$

$$= \widetilde{I}(\mathbf{u}[1-1/N])\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1} \tag{13}$$

which is independent of j and $\mathbf{c}_k$.

The evaluation of the sum over j can, therefore, be performed straightforwardly giving:

$$\begin{aligned}\widetilde{D}^c_k(\mathbf{u},\mathbf{c}_k) &= \left\langle \widetilde{D}^c_k(\mathbf{u},\mathbf{x}_1,...,\mathbf{x}_N)\right\rangle_{\mathbf{x}_1,...,\mathbf{x}_N}\\ &= \sum_{j=1}^{N}\{\widetilde{I}(\mathbf{u}[1-1/N])\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1}\}\\ &= N\,\widetilde{I}(\mathbf{u}[1-1/N])\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1}\end{aligned} \tag{14}$$

The result in (14) is independent of $\mathbf{c}_k$ and hence:

$$\begin{aligned}\widetilde{D}^c(\mathbf{u}) &= \left\langle \widetilde{D}^c_k(\mathbf{u},\mathbf{c}_k)\right\rangle_{\mathbf{c}_k}\\ &= N\,\widetilde{I}(\mathbf{u}[1-1/N])\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1}\end{aligned} \tag{15}$$

It is convenient to introduce a normalized spectrum $\widetilde{Q}_N(\mathbf{u})$ defined as

$$\widetilde{Q}_N(\mathbf{u}) \equiv \frac{D^c(\mathbf{u})}{N} \qquad \text{,yielding the important result}$$

$$\widetilde{Q}_N(\mathbf{u}) = \widetilde{I}(\mathbf{u}[1-1/N])\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1} \tag{16}$$

Expression (16) is a relationship between the normalized spectrum of the stationary image $\widetilde{I}(\mathbf{u})$ and the spectrum $\widetilde{Q}_N(\mathbf{u})$ of the centroided average of those frames containing exactly N photons/frame. The independence of (13) on $\mathbf{c}_k$ shows that $\widetilde{Q}_N(\mathbf{u})$ is a translating invariant quantity that depends only on the spectrum $\widetilde{I}$ and on the number of photons/frame N. As a consequence the quantity $\widetilde{Q}_N(\mathbf{u})$ is the same either for the randomly moving image as well as for the stationary image.

Equation (16) can be re-expressed in object space:

$$\begin{aligned}Q_N(\mathbf{r}) &= \mathcal{F}^{-1}\{\widetilde{Q}_N(\mathbf{u})\}\\ &= \mathcal{F}^{-1}\{\widetilde{I}(\frac{N-1}{N}\mathbf{u})\quad \left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1}\}\\ &= \mathcal{F}^{-1}\{\widetilde{I}(\frac{N-1}{N}\mathbf{u})\}\quad\otimes\quad \mathcal{F}^{-1}\{\left[\widetilde{I}(-\mathbf{u}/N)\right]^{N-1}\}\\ &= \mathcal{F}^{-1}\{\widetilde{I}(\frac{N-1}{N}\mathbf{u})\}\otimes\\ &\qquad \otimes\underbrace{\mathcal{F}^{-1}\{\widetilde{I}(-\mathbf{u}/N)\}\otimes.......\otimes\mathcal{F}^{-1}\{\widetilde{I}(-\mathbf{u}/N)\}}_{(N-1)\quad terms}\end{aligned} \tag{17}$$

An examination of the above results for some particular values of N are instructive. For instance for $N = 2$ Eq.(16) reduces to

$$\begin{aligned}\tilde{Q}_2(\mathbf{u}) &= \tilde{I}(\frac{\mathbf{u}}{2}) \quad \tilde{I}(-\frac{\mathbf{u}}{2}) \\ &= \tilde{I}(\frac{\mathbf{u}}{2}) \quad \tilde{I}^*(\frac{\mathbf{u}}{2}) \\ &= \left|\tilde{I}(\frac{\mathbf{u}}{2})\right|^2 \end{aligned} \tag{18}$$

which is the power spectrum of $\tilde{I}(\mathbf{u})$ where the variable $\mathbf{u}$ is scaled by a factor of 2. Fourier transforming $\tilde{Q}_2(\mathbf{u})$ yields the scaled autocorrelation of the image $I(\mathbf{r})$.

For N=3

$$\begin{aligned}\tilde{Q}_3(\mathbf{u}) &= \tilde{I}(\frac{2}{3}\mathbf{u}) \quad \left[\tilde{I}(-\frac{\mathbf{u}}{3})\right]^2 \\ &= \tilde{I}(\frac{2}{3}\mathbf{u}) \quad \tilde{I}(-\frac{\mathbf{u}}{3})\tilde{I}(-\frac{\mathbf{u}}{3}) \end{aligned} \tag{19}$$

The above equation is related to the triple correlation[7-8] $I^{(3)}(\mathbf{x}_1, \mathbf{x}_2)$ defined for real functions as

$$I^{(3)}(\mathbf{x}_1, \mathbf{x}_2) = \int_{-\infty}^{+\infty} I(\mathbf{x})\, I(\mathbf{x}+\mathbf{x}_1)\, I(\mathbf{x}+\mathbf{x}_2)\, d\mathbf{x} \tag{20}$$

and its Fourier transform $\tilde{I}^{(3)}(\mathbf{u}, \mathbf{v})$ known as the bispectrum

$$\tilde{I}^{(3)}(\mathbf{u}, \mathbf{v}) = \tilde{I}(\mathbf{u})\, \tilde{I}(\mathbf{v})\, \tilde{I}(-\mathbf{u}-\mathbf{v}) \tag{21}$$

It must be pointed out that for $\mathbf{u} = \frac{2}{3}\mathbf{u}$, $\mathbf{v} = -\frac{1}{3}\mathbf{u}$ or $\mathbf{u} = -\frac{1}{3}\mathbf{u}$, $\mathbf{v} = \frac{2}{3}\mathbf{u}$, which are representations of lines in the $\mathbf{u}, \mathbf{v}$ plane, equation (21) reduces to (16).

Another limiting case is when $N \to \infty$. Remembering that $\tilde{I}(\mathbf{u})$ has been normalized to unit at $\mathbf{u} = 0$ it can be seen that:

$$\begin{aligned}\lim_{N\to\infty} \tilde{Q}_N(\mathbf{u}) &= \lim_{N\to\infty} \left\{\tilde{I}(\mathbf{u}[1-1/N])\, \tilde{I}(-\frac{\mathbf{u}}{N})^{N-1}\right\} \\ &= \lim_{N\to\infty} \left\{\tilde{I}(\mathbf{u}[1-1/N])\right\} \times \lim_{N\to\infty} \left\{\tilde{I}(-\frac{\mathbf{u}}{N})^{N-1}\right\} \\ &= \tilde{I}(\mathbf{u}) \end{aligned} \tag{22}$$

which is the normalized stationary image spectrum.

3 - CENTROIDING PHOTON DATA

Experiments were carried out using simulated one dimensional photon data generated by a computer where the desired low-light level is achieved by selecting the value of the Poisson mean ($\overline{N}$) in the Poisson distribution of photons detected in each frame interval. In the particular simulation experiment described in this paper, we set $\overline{N} = 3$ photons/frame. Frames with 0 or 1 photon/frame are disregarded, because they do not carry any information concerning the intensity distribution of the image, and a total of 80,084 frames of a randomly translating image (of a binary star) containing $N \geq 3$ photons/frame were generated. These frames can be grouped in sets of frames containing a number of photons ranging from N=3 up to N=13 photons/frame in this case. The actual distribution for each value of N is shown in Table 1 for the photon-data used in this simulation.

For each particular set of N-photon frames, one should centroid the corresponding M_N frames, average(add) and divide by N to find the normalized image estimator $Q_N(\mathbf{r})$ and its Fourier transform $\tilde{Q}_N(\mathbf{u})$:

$$Q_N(\mathbf{r}) = \frac{1}{M_N N} \sum_{1}^{M_N} \{\text{ Frames with N photons/frame}\} \tag{23}$$

and

$$\widetilde{Q}_N(\mathbf{u}) = \mathcal{F}\{Q_N(\mathbf{r})\} \tag{24}$$

Figure 2 depicts $Q_N(\mathbf{r})$ for N=2 up to N=13 photons/frame. For N=2 it can be seen that $Q_2(\mathbf{r})$ exhibits the same shape as the autocorrelation of the binary but distributed in a smaller (shrinked) region of space. It can also be seen that the average $Q_N(\mathbf{r})$ becomes more and more similar to the stationary image as N increases.

Frames containing more photons are bound to carry more information concerning the image than frames with a lesser number of photons. The limiting cases are those with no information content (N=0) and those with the image itself ($N \to \infty$). For instance frames with N=2 only carry information concerning the modulus of all possible distances within the stationary image whereas frames with N=3 photons/frame besides the modulus also carry information concerning the relative positions of points in the object giving therefore information about its structure. This fact allows one to use frames with at least N=3 photons/frame to reconstruct the image through the technique of triple correlation.

Accepting this idea one should give increased weight to those frames containing more photons. A possible way of implementing this is to construct an image estimator by adding, <u>with equal weight</u>, the quantities $Q_N(\mathbf{r})$ i.e. :

$$I^c_{weighted} = \frac{1}{\{\text{number of quantities } Q_N(\mathbf{r})\}} \sum_{N_{min}}^{N_{max}} Q_N(\mathbf{r}) \tag{25}$$

In this particular case the 12 quantities, $Q_2(\mathbf{r})$, $Q_3(\mathbf{r})$,........, $Q_{13}(\mathbf{r})$, so formed are superimposed(averaged) forming the image estimator, $I^c_{weighted}$:

$$I^c_{weighted} = \frac{1}{12} \sum_{N=2}^{13} Q_N(\mathbf{r}) \tag{26}$$

The above procedure should be compared with the usual centroiding procedure. Let the total number of frames be $\mathcal{M} = \sum_N M_N$. The usual centroiding procedure would consist of averaging all centroided frames irrespective of the number of photons per frame N i.e.:

$$I^c_{usual} = \left\{\frac{1}{\mathcal{M}\mathcal{N}}\right\} \sum_{\text{all frames}} \{\text{all frames irrespective of N value}\} \tag{27}$$

where $I^c_{usual}(\mathbf{r})$ is the usual centroided image estimate and $\mathcal{N} = \sum_{N=2}^{N=N_{max}} M_N\, N$ is the total number of photons of all $\mathcal{M}$ frames.

Figure 3b shows the result using the procedure given by Eq. (27), $I^c_{usual}(\mathbf{r})$, whereas Figure 3c shows $I^c_{weighted}$ obtained according to the procedure described by Eq. (25). Although $I^c_{weighted}$ is an improved estimate of the image further work has do be done concerning the way the quantities $Q_2(\mathbf{r})$, ..., $Q_N(\mathbf{r})$ should be added to form $I^c_{weighted}$ because, in the present case, although each frame used to build $Q_N(\mathbf{r})$, when N is larger, contains more structural information than a frame used to build $Q_N(\mathbf{r})$, when N is smaller, the <u>number</u> of frames with large N is smaller than the number of frames with small N.

4 - IMAGE RETRIEVAL FROM ONE-DIMENSIONAL CENTROIDED IMAGES - $Q_N(x)$

Applying Eq. (16) for real one dimensional images ($\widetilde{I}(-u) = \widetilde{I}^*(u)$) and changing $\frac{u'}{N} \to u$ one gets:

$$\widetilde{Q}_N(N\,u) = \widetilde{I}([N-1]u) \quad \left[\widetilde{I}^*(u)\right]^{N-1} \tag{28}$$

The randomly translating image is sampled at $\mathcal{L}$ bins equally spaced in the focal plane and the Fourier transforms $\widetilde{Q}_N(u)$ is therefore also sampled at $\mathcal{L}$ bins spaced by Δu. Each bin frequency can then be described as $u = k\,\Delta u$ where k = 0, 1, 2, ...,($\mathcal{L}$-1) allowing each sampled frequency to be described by its k value.

As $\widetilde{Q}_N(Nk)$, $\widetilde{I}([N-1]k)$ and $\widetilde{I}(k)$ are complex quantities they can be written as:

$$\widetilde{Q}_N(Nk) = \left|\widetilde{Q}_N(Nk)\right| \, exp\,\{i\,\Theta_N(Nk)\} \tag{29}$$

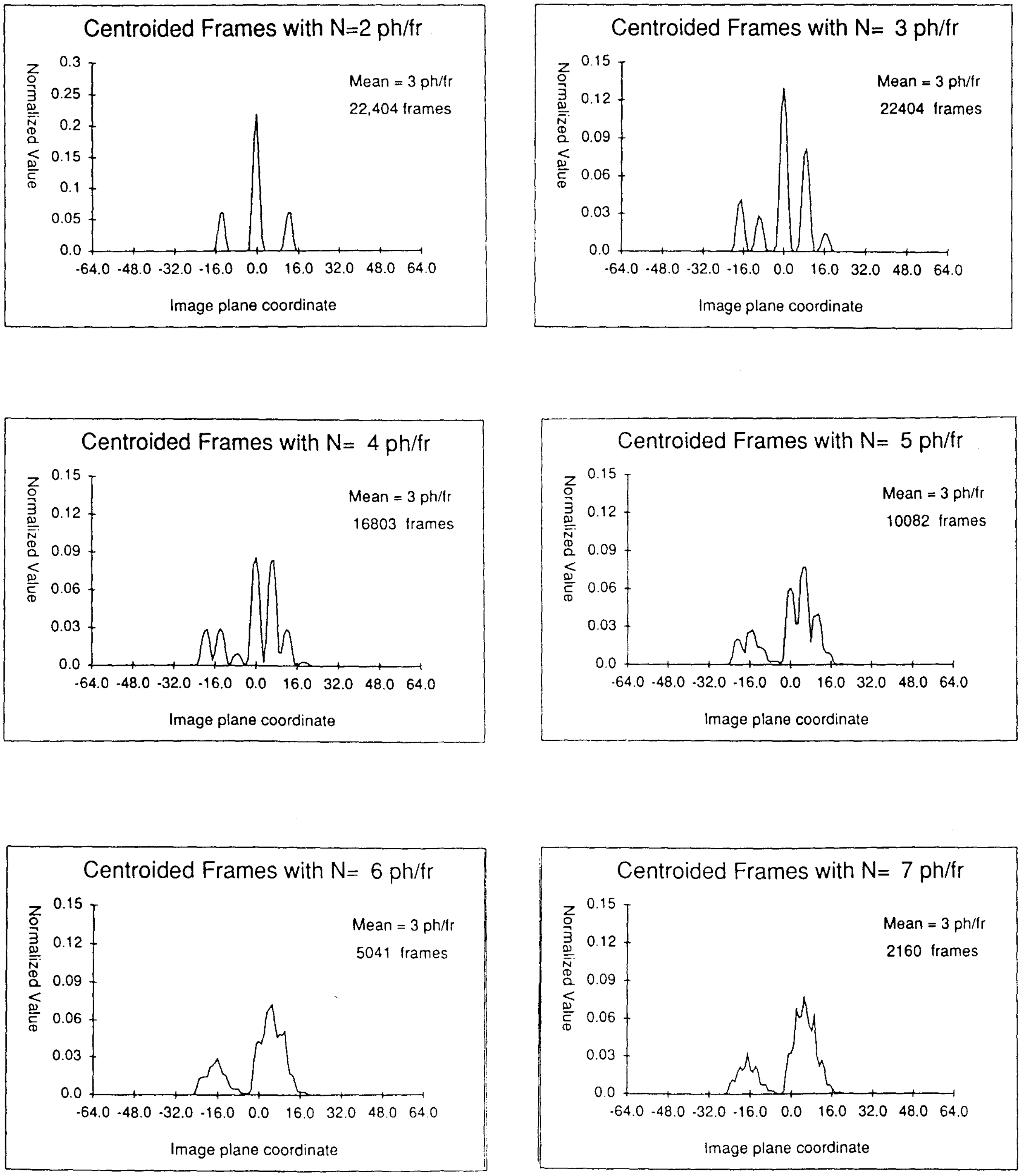

Figure 2 - Averaged Centroided Image Estimators $Q_N^{(r)}$

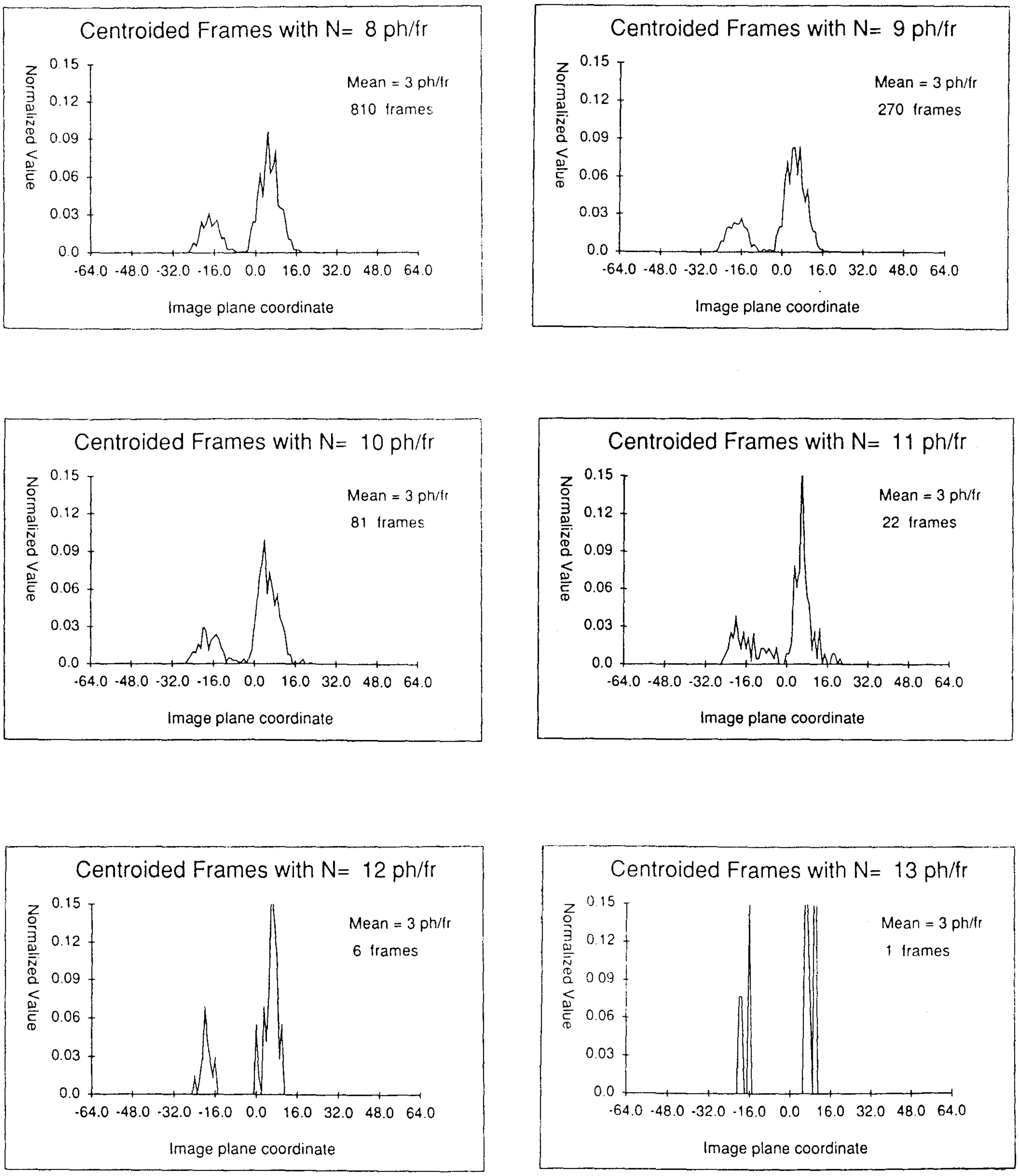

Figure 2 - continued

N	M_N	N	M_N	N	M_N
2	22404	6	5041	10	81
3	22404	7	2160	11	22
4	16803	8	810	12	6
5	10082	9	270	13	1

Table 1 - Number of frames, M_N, containing N photons/frame

$$\widetilde{I}([N-1]k) = \left|\widetilde{I}([N-1]k)\right| \, exp\{i\,\phi([N-1]k)\} \tag{30}$$

$$\widetilde{I}^*(k) = \left|\widetilde{I}^*(k)\right| \, exp\{-i\,\phi(k)\} \tag{31}$$

Substituting in (28):

$$\frac{\left|\widetilde{Q}_N(Nk)\right|}{\left|\widetilde{I}([N-1]k)\right| \left|\widetilde{I}^*(k)\right|} = e^{i\{\phi([N-1]k) - (N-1)\phi(k) - \Theta_N(Nk)\}} \tag{32}$$

The left-hand-side being real $\Rightarrow$

$$\Theta_N(Nk) = \phi([N-1]k) - (N-1)\phi(k) \tag{33}$$

Equation (31) is a recurrence relation linking phases at frequency k and $[N-1]k$ through the phase of the average quantity $\widetilde{Q}_N(Nk)$ and it can be seen that for N=2

$$\begin{aligned}\Theta_2(2k) &= \phi(k) - \phi(k) \\ &= 0\end{aligned} \tag{34}$$

and therefore no information about the phase can be retrieved. But for N=3 Eq.(31) becomes:

$$\Theta_3(3k) = \phi(2k) - 2\,\phi(k) \tag{35}$$

and hence from phase at frequency k one can reconstruct the phase at 2k up to $3k \leq \mathcal{L}/2 - 1$ where $\mathcal{L}$ is the actual number of bins used to sample $Q_3(\mathbf{r})$.

The method can be better understood through an example in which a one dimensional image is sampled at 32 points and therefore $\widetilde{Q}_N(Nk)$ is determined only at 32 frequency bins. Due to the fact that $\widetilde{I}(-k) = \widetilde{I}^*(u)$ one has to find the phases only in half the total number of bins and then reverse their sign. In this particular example one has, therefore, to consider only 16 frequency bins (Figure 4).

For k=0 the phase $\phi(0) = 0$. As $\widetilde{Q}_N(Nk)$ is shift-invariant one can always determine $\widetilde{I}(k)$ apart from a constant phase shifting factor. By appropriately choosing this factor one can always set the phase at $\phi(1) = 0$. From then on by successcively applying the recurrence relations (33), for each $\widetilde{Q}_N(Nk)$, one reconstructs all the phases by using averages of frames with N up to 16.

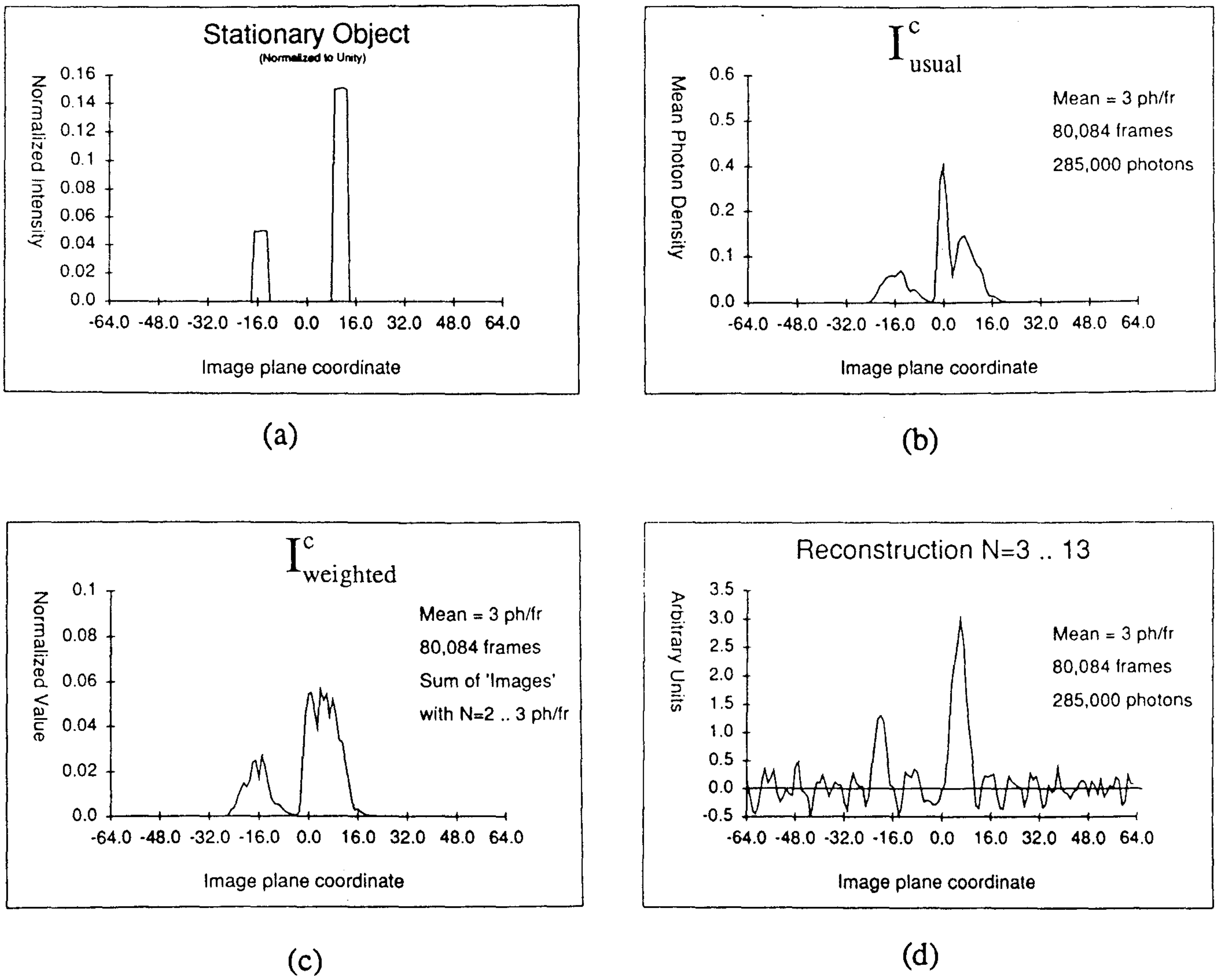

Figure 3 - Comparison between the object,

I^c_{usual} , $I^c_{weighted}$ and the Reconstruction.

Using N=3 ph/fr and departing from $\phi(1)$ one gets the phase at 2 using $\Theta_3(3)$; from $\phi(2)$ one gets $\phi(4)$ using $\Theta_3(6)$; from $\phi(4)$ one gets $\phi(8)$ using $\Theta_3(12)$ but from $\phi(8)$ this recurrence relation can not be applied anymore because it would give us the phase at $\phi(16)$, which does not exist, through the use of the phase $\Theta_3(24)$ which also does not exist.

For N=4 ph/fr the recurrence relation (33) becomes:

$$\Theta_4(4k) = \phi(3k) - 3\,\phi(k) \tag{36}$$

Departing from $\phi(1)$ and using the same procedure described for the case of N=3 ph/fr one finds $\phi(3)$, $\phi(6)$ and $\phi(9)$.

In general, for a frequency bin k, where k is a prime number, one can only reconstruct its phase by using an average $\widetilde{Q}_N(Nk)$ such that N=k+1 photons/frame. In this particular example to reconstruct the phases up to bin 15 one needs averages of frames with N up to 16 photons/frame. The modulus of $\widetilde{I}(k)$ is found from the Fourier transform of the auto correlation function, the power spectrum $\left|\widetilde{I}(u)\right|^2$.

Figure 3d shows the reconstructed image obtained using this method with $\mathcal{L} = 128$ bins and using frames with N only up to 13 ph/frame i.e. not all the frequencies were reconstructed. Despite the fact that no other image processing technique has been used, e.g. enforcing positivity, Fig. 3d, nevertheless, gives a good estimate of the image and in this particular case also a good estimate of the relative brightness of the stars.

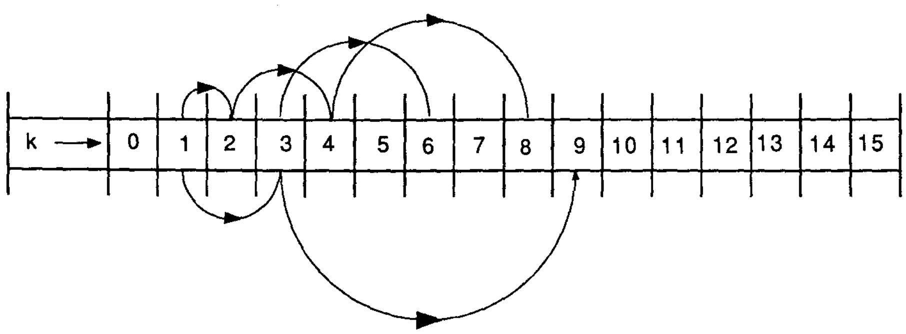

Figure 4 - Pictorial representation of the phase reconstruction algorithm

5 - CONCLUSIONS AND FUTURE WORK

The results show that centroiding, although a very simple and computationally fast technique, can be used to retrieve an image of a randomly translating object at very low light level.

Future work could address the following:

(i) how to combine the $Q_N(\mathbf{r})$ in an optimum way;
(ii) extend the phase retrieval algorithm to two dimensions and
(iii) apply the technique to speckled imaging.

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7 - REFERENCES

1 - C.R. Lynds, S.P. Worden and J.W. Harvey, Ap. J., **207**, 174(1976).

2 - S.P. Worden, C.R. Lynds and J.W. Harvey, JOSA, **66**, 1243(1976).

3 - R.H.T. Bates, M.O. Milner, G.I. Lund and A.D. Seagar, Opt. Comm., **26**, 22(1978).

4 - R.H.T. Bates and F.M. Cady, Opt. Comm., **32**, 365(1980).

5 - F.M. Cady and R.H.T. Bates, Opt. Lett., **5**, 438(1980).

6 - J.W. Goodman and J.F. Belsher, "*Photon Limited Images and Their Restoration*" Technical Report RADC-TR-76-50 (March 1976); "*Precompensation and Postcompensation of Photon Limited Degraded Images*", Technical Report RADC-TR-76-382 (December 1976); "*Photon Limitations in Imaging and Image Restoration*", Technical Report RADC-TR-77-175 (May 1977) (all available from Rome Air Development Center, Griffiss AFB, NY 13441.

7 - A.W. Lohmann, G. Weigelt and B. Wirnitzer, Appl. Opt., **22**, 4028(1983).

8 - A.W. Lohmann and B. Wirnitzer, Proc. IEEE, **72**, 889(1984).