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LEAST-SQUARES RECONSTRUCTION OF THE OBJECT PHASE FROM THE BISPECTRUM

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A least-squares method for the reconstruction of the object phase from the phase of the bispectrum is presented. The method is successfully applied to experimental photon-limited data in the visible and to an extended infra red object. A comparison to other least-squares methods is given.

1. INTRODUCTION

The bispectrum has been shown to be a useful tool for astronomical imaging through turbulence[1-4]. Once the average bispectrum of a number of short-exposure images has been computed and corrected for artefacts, the problem is then to find a satisfactory method for calculating the Fourier phase of the object spectrum from the phase of the bispectrum.

Usually a recursive method has been used to recover the object phase[2]. The disadvantage of this method lies in the fact that the error in the bispectral phase for low spatial frequencies affects the recovered object phases at higher frequencies. That means that since the object phase of a high spatial frequency is the sum of the preceding bispectral phases, possibly a couple of hundred values, the errors of all these phases are summed up too.

Applying a least-squares method promises an advantage in the respect that the errors of the bispectral phases are not accumulated for the calculation of each new object phase[5]. The relation between the bispectral phase and the reconstructed object phases reads as

$$\Delta_{i,j} = \beta_{i,j} - (\phi_i + \phi_j - \phi_{i+j}), \qquad (1)$$

where $\Delta_{i,j}$ is the difference between the measured average bispectral phase $\beta_{i,j}$ and the reconstructed object phase ϕ at discrete points i, j and $i + j$.

Solving (1) for all bispectral points by a least-squares method is not completely straightforward because the average bispectrum phase is calculated only over the interval (-π to +π), whilst the bispectrum phase obtained from the reconstructed object phase lies within (-3π, +3π). A classical least-squares method, when used to solve the above overdetermined system of equations fails to incorporate the 2π phase ambiguity.

In the following we shall give a brief introduction to least-squares algorithms and present three different methods[6-11] to overcome the phase ambiguity. Finally the successful application of a least-squares method to photon-limited data in the visible and to infra red data will be presented.

2. CLASSICAL FORMULATION

The classic least-squares problem relies on estimating N unknown constants from M equations, where $M > N$. In the context of retrieving the object phase from the estimated bispectrum phase, the object phases are unknown but assumed constant, and the bispectral points are assumed to be noisy but unbiased estimates of the true bispectrum. The noise on each bispectral point is assumed to be independent of the noise

on the other bispectral points.

Assembling all the equations (1) associated with the bispectrum produces the following matrix formulation,

$$\Delta = \beta - A\phi, \quad (2)$$

where Δ is the difference vector, β is a vector comprising the M known bispectral phases, A is an MxN sparse matrix, and ϕ is a vector containing the N unknown object phases. As every bispectral phase is built up of three object phases the matrix A contains only three non-zero elements in each row. As not all bispectrum values have equal amounts of noise a weighting factor W is introduced such that the appropriate error measure to minimise is,

$$E = (\beta - A\phi)^T W(\beta - A\phi), \quad (3)$$

where a superscript T is used to indicate matrix transpose. W is an MxM diagonal matrix, with the ith element given by the reciprocal of the variance of the bispectral point. The matrix W is diagonal as a result of the assumed independence of the bispectrum errors.

The form of (3), when phase wrapping is ignored, is a parabolic error surface with a single global minimum. This minimum can be found by differentiating the error function and setting it to zero,

$$\begin{aligned} \frac{\partial E}{\partial \phi} &= -2(A^T\beta - A^TA\phi)W \\ &= 0. \end{aligned} \quad (4)$$

The problem that arises in recovering the true object phase from the bispectrum is that the object phase is not defined continuously on the interval $(-\infty, \infty)$. In addition to this even the perfect reconstruction of ϕ can result in differences Δ of + or -2π . It is then possible that there exists a solution with a lower error metric than the solution obtained by (4) and there is no longer a guaranteed single minimum of the error function.

3. UNWRAPPING METHODS

3.1. Adding multiples of $\pm 2\pi$

Here multiples $k_{i,j}$ of 2π are added to the bispectral phase $\beta_{i,j}$ to remove the effect of the 2π ambiguity in (3). Then a satisfactory image recovery is possible[6,7]. The difference $\Delta_{i,j}$ becomes

$$\Delta_{i,j} = \beta_{i,j} - (\phi_i + \phi_j - \phi_{i+j}) + k_{i,j}2\pi. \quad (5)$$

The problem with this approach is that the unwrapping vector is computed from the noisy bispectrum resulting in incorrect values for $k_{i,j}$. Thus, the solution produced by the least-squares algorithm is not necessarily optimal. An application of this method to measured data has not yet been published.

3.2. The Phasor Approach

The second approach to remove the 2π ambiguity is to take the difference of the phasors rather than of the phases[8–10],

$$\Delta^p_{i,j} = \exp(i\beta_{i,j}) - \exp(i(\phi_i + \phi_j - \phi_{i+j})). \quad (6)$$

Meng et al[8] have done this but use a different technique for solving the least-squares problem. As weighting function W they take the square of the modulus of the bispectrum over the variance of the modulus. They solve (4) for ϕ, getting

$$\phi = (A^TA)^{-1}A^T\beta. \quad (7)$$

Since this solution for ϕ should also result in a least-squares estimate of ϕ, they attempt to solve these equations with a relaxation technique.

Matson's[9] method of phasors changes the weighting applied by Meng et al[8] to those equations in which the bispectrum point is a function of two times an object phase still using the variance of the modulus of the bispectrum.

Gorham et al[10] simply use phasors in the final differencing between the measured and estimated bispectrum phase and minimise the weighted sum of the squared difference phasor's lengths. For those bispectral equations where the difference between the estimated and measured bispectrum is small the contribution to the computed error metric (3) should be almost the same whether phasor or phase differences defined modulo 2π are used.

3.3. The Direct Method

A logical alternative to the phasor method is to take the phase difference $\Delta_{i,j}$ in equation (1) modulo 2π. This method has been proposed by Haniff[11] who presented results for one-dimensional simulated data. The cost function (3) is redefined to incorporate the modulo 2π phase wraps and is

$$\begin{aligned} E &= mod_{2\pi}(\beta - A\phi)^T W mod_{2\pi}(\beta - A\phi) \quad (8) \\ &= \sum_{i,j} W_{i,j} mod_{2\pi}(\beta_{i,j} - (\phi_i + \phi_j - \phi_{i+j}))^2. \end{aligned}$$

The weighting function W is now defined as the reciprocal of the variance of the *bispectrum phase* β,

$$\begin{aligned} W_{i,j} &= [Var(\beta_{i,j})]^{-1} \\ &= \left[\frac{Var_{\perp}(B_{i,j})}{|B_{i,j}|^2}\right]^{-1}, \quad (9) \end{aligned}$$

with $Var_{\perp}(B_{i,j})$ the variance in the direction perpendicular to the direction of the complex bispectrum phasor $B_{i,j}$ and $|B_{i,j}|^2$ the square of the bispectral modulus.

The incorporation of modulo 2π phase wraps introduces a non-linearity into the calculation such that several local minima can occur in the numerical calculation before the desired global minimum is found. This non-linearity can be reduced and computing time decreased by restricting the difference removed by the modulo function to $\pm 4\pi$, because $\beta_{i,j}$ cannot be smaller than $-\pi$ and the sum of ϕ not greater than 3π. We have coded this algorithm for the two-dimensional case, using a least-squares routine (E04DGF) from the NAG library.

The image quality of our reconstructions improved consistently with the increasing number of bispectral values. This is in contrast to other investigations (e.g. 8) but in accordance with the general behaviour of least-squares algorithms.

4. RESULTS

Infra red data of an extended object (IRC +10216, courtesy of J. Christou and C. Haniff) and photon limited data of several binary stars (courtesy of E. Hege) have been used to investigate the least-squares method described in section 3.3.

The bispectrum of an array with NxN elements has of the order of $N^4/12$ non redundant elements. Thus the bispectrum of an array with 64x64 elements contains about 10^6 complex values. In order to avoid difficulties with the storage capability of the computer this amount of data should be reduced. Because in speckle interferometry only bispectral elements for spatial frequencies i or j smaller than the frequency related to the Fried parameter have a high signal to noise ratio[3], one starts taking these elements of the bispectrum. A group of bispectral points $B_{i_0,j}$ with a fixed spatial frequency i_0 and all possible values for j is called a subplane of the bispectrum. In the two-dimensional case a subplane has a high SNR if the spatial frequency vector related to the index i_0 has a length smaller than the frequency vector related to the Fried parameter. The maximal number of subplanes used for our reconstructions is 80, which is 1/20 of all subplanes in a two-dimensional array with a diameter of 64 pixels. The highest spatial frequency i_0 for this case is about 1/3 of the diffraction limit of the telescope. In most cases this frequency is beyond the limit given by the Fried parameter, so that bispectral values with a low SNR are also taken.

The average computing time for our least-squares algorithm on a SUN-Sparcstation is 7 minutes when using 10 subplanes and 25 minutes with 80 subplanes. The average number of iterations is 100. The initial estimate for the phase can be zero or a random number between + and $-\pi/2$, producing always the same results. A random number of a wider interval introduces a large linear term in the phase reconstruction shifting the reconstructed image to the border of the array and decreasing the image quality a little bit.

The modulus of the object spectrum is recovered after taking the square root of the quotient of the power spectrum of the object over the power spectrum of the reference star. The necessary windowing is always performed with the telescope MTF. As this is a very important point we will

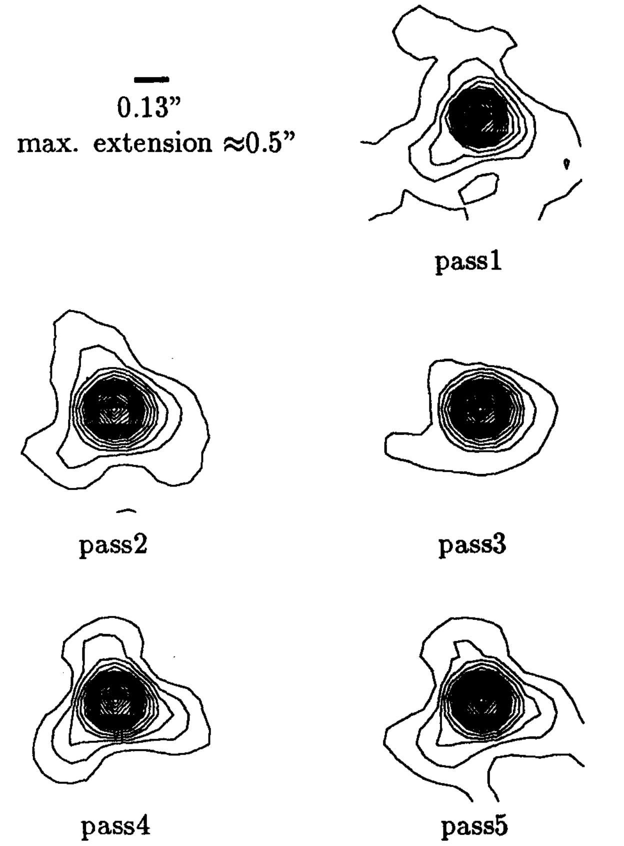

Figure 1: Reconstruction of IRC +10216 for five different passes. 80 subplanes are used for the phase reconstruction and 20 contours are displayed. The diffraction limit of the telescope is 0.13".

return to it in section 5.

The infra red data of IRC +10216 was taken at $\lambda = 2.2\ \mu m$ on a 4m telescope (diffraction limit of resolution 0.13") with the NOAO 2-D IR speckle camera in May 1989. Five passes of approximately 200 frames each were recorded together with three passes for the reference star 31 Leo. Fig. 1 shows the reconstructed image intensities for five different passes. There is no significant difference compared to other reconstructions on the same data averaging over all five passes[12,13]. The variation between different passes is greater than between different reconstruction methods. Possibly the change of statistical parameters of the atmosphere between passes for the object and reference star is responsible for this variation. A similar variety in the reconstruction occured when phase retrieval on the power spectrum was applied. It has been suggested that the triangular shape of the reconstruction could in part be caused by aberrations of the telescope. However, this structure was not apparent in our reconstructions of the reference star.

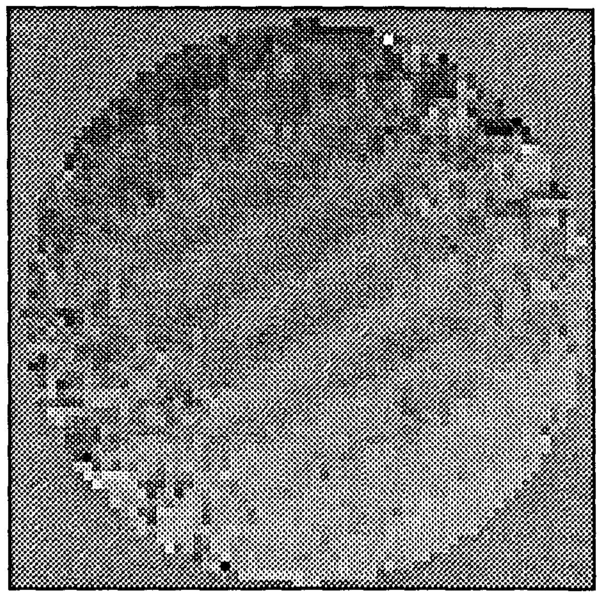

Figure 2: The reconstructed phase of 126 Tau using 60 subplanes for the reconstruction.

The photon limited data was recorded at $\lambda = 550$ nm on a 2.3 m telescope (diffraction limit 0.06") on a 256x256 array. The phase is reconstructed for the central 64x64 part. An average number of 500 photons/frame and 5000 frames were registered. Fig. 2 shows the reconstructed phase for 126 Tau and Fig. 3 shows the reconstructed image intensities for three binaries β Del, μ Ori and 126 Tau.

The negative parts of the intensity give an indication for the noise in the reconstruction. They disappear if a narrower window function is taken (see Fig. 3d for 126 Tau). The third 'star' in the β Del reconstruction is an artefact probably caused by a lack of contrast in the reconstructed phase.

a)

b)

c)

d)

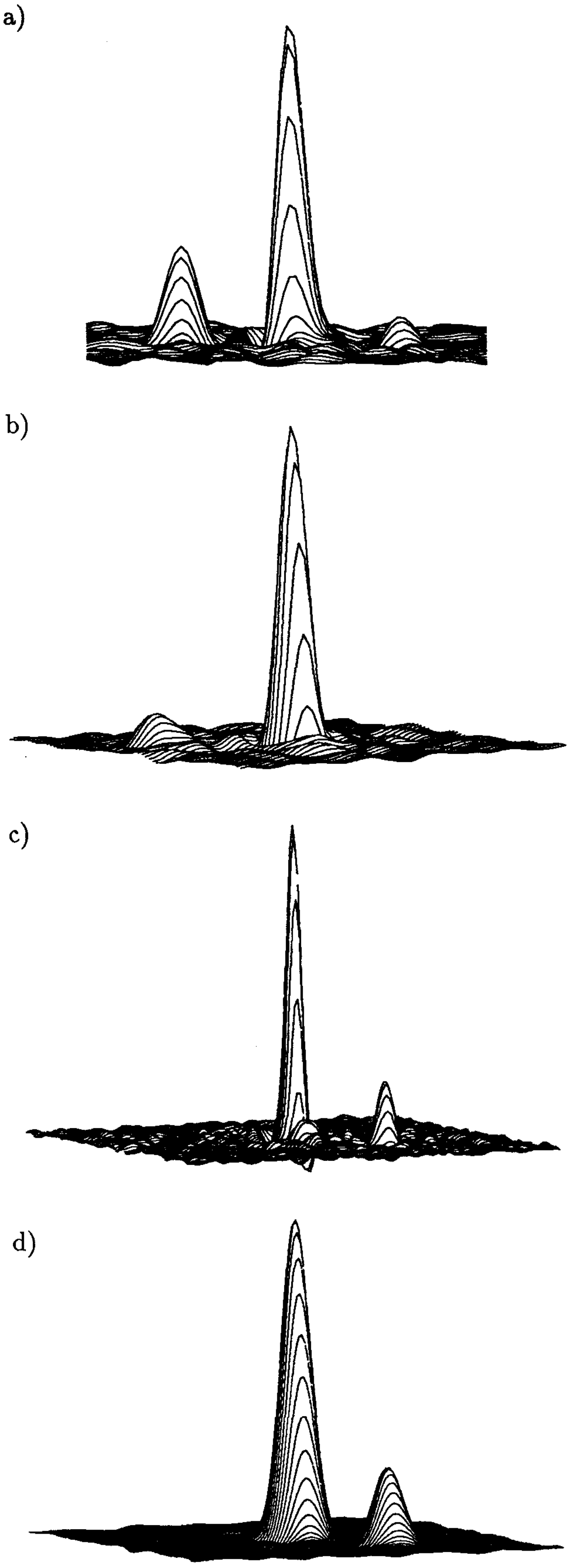

Figure 3: Reconstructions of β Del, μ Ori and 126 Tau. For β Del and μ Ori the central 64x64 part of the array is shown, for 126 Tau it is the central 128x128 part. For the reconstructions shown in (a), (b) and (c) the telescope MTF has been taken as window function. For the reconstruction in (d) a Gaussian function with the value 1/e at 60 % of the diffraction limit has been taken. In each figure the negative parts can be seen, giving an indication for the noise of the reconstruction.

If the phase is set to zero, i.e. the fringe pattern has zero contrast, the reconstructed image is similar to its autocorrelation having two small peaks symmetrical to the high peak at the center. The transition from the fringe pattern with zero contrast to higher contrast causes one of the smaller peaks to disappear. This is what happens for μ Ori and 126 Tau. It is not clear why the β Del reconstruction shows this artefact.

5. DISCUSSION

Fig. 4 shows the improving quality of the reconstruction of 126 Tau when the number of subplanes is increased. Surprisingly, the reconstructed phases do not show any improvement. Even a scan through the center of the array (Fig. 5) does not reveal a higher contrast in the reconstructed fringe pattern or anything else that could easily be connected to improved quality. Thus, the only measure for a good or a bad phase reconstruction is the reconstructed image intensity.

Matson[9] used 40 subplanes for his reconstruction of 126 Tau and truncated the spectrum at 60% of the diffraction limit. The reconstruction is very similar to our reconstruction (Fig. 4) except that truncating the spectrum, i.e. multiplying it with a circular pupil function causes side lobes around the brighter star. Since multiplication in frequency space is eqivalent to convolution in image space, the reconstructed image intensity is convolved with a besinc function, the Fourier transform of the circular function. The negative parts of the besinc function cause negative parts in the image intensity regardless of the quality of the phase reconstruction. One should rather use

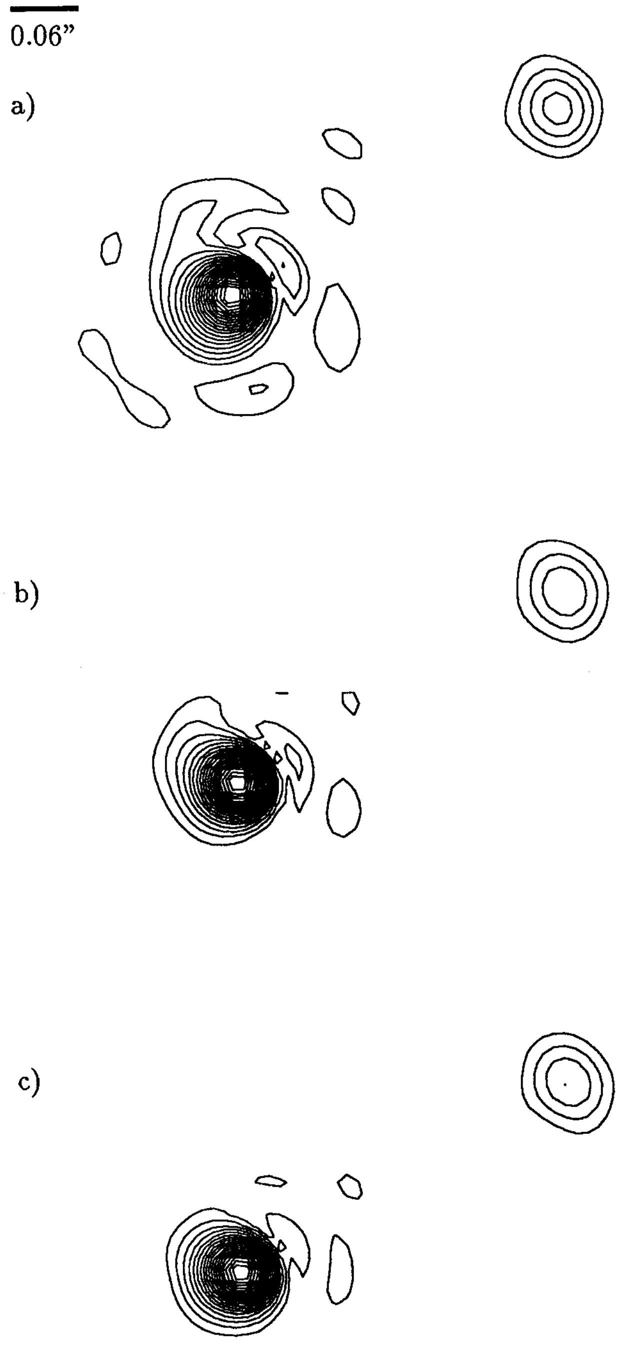

Figure 4: Reconstruction of 126 Tau for different numbers of subplanes. 4900 frames with an average of 610 photons per frame are used. Thick lines indicate negative values and 20 contours are displayed. The diffraction limit of the telescope is 0.06”. For the reconstruction shown in (a) 10 subplanes are used, CPU time is 6 minutes and the number of iterations 108. For (b) the numbers are 28 subplanes, 25 minutes and 181 iterations and for (c) 60 subplanes, 23 minutes and 79 iterations.

a function like the MTF with a positive Fourier transform to window the spectrum.

Meng et al[8] windowed the spectrum for their reconstructions of β Del and μ Ori with the telescope aperture function, probably the MTF, with a cutoff frequency at 65% of the diffraction limit. The quality of their reconstruction is remarkably good given that only two subplanes of the bispectrum were used.

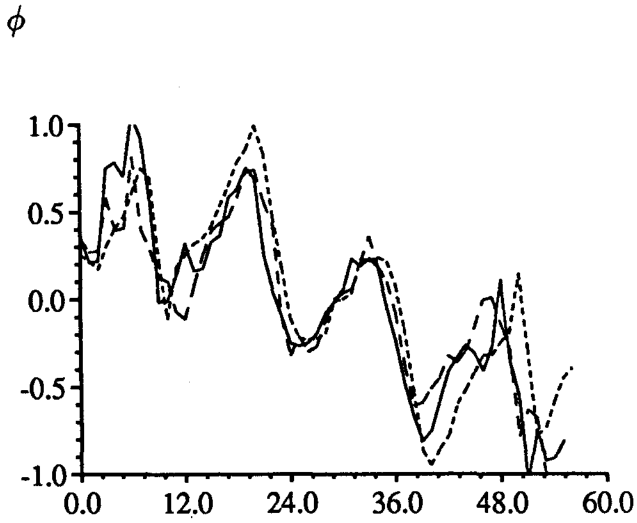

Figure 5: Scans through the central part of the reconstructed phase ϕ of β Del for different number of subplanes. There is no observable change in fringe contrast.

The influence of the window function is more severe than only producing more or less smooth reconstructions. Observing double stars one is interested in their position and the relative brightness. As the fringe pattern in the phase and the modulus of the object spectrum is not reconstructed perfectly up to the diffraction limit but becomes noisier for high frequencies (see Fig. 2) a wider window function eventually adds only noise to the spectrum. Fourier transforming the resulting spectrum one yields a higher peak at the centre then using a narrower window function that includes fewer noisy values of the spectrum. Thus the relative brightness depends on the width of the window function. For instance the relative brightness of the darker star of β Del is .31 if the telescope MTF is used and .37 if an MTF with a

cutoff frequency at 65% of the diffraction limit is used. The problem is that the window function cannot be defined properly because the noise increases continously.

The choice of the telescope MTF appears to be the most logical choice of window function. This would allow easy comparison of the results from different reconstructions.

6. CONCLUSIONS

The direct least-squares method proved to be a logically simple and reliable method. The algorithm is not sensitive to the initial estimate of the object phase and always produced a solution.

The advantage of the least-squares method over the recursive method is that the errors in the bispectral phases are not recursively accumulated when estimating the phases of higher spatial frequencies of the object spectrum. Since these higher frequencies determine the resolution in the reconstruction, it is important that these are estimated accurately. Least-squares estimation results in better phase and image reconstructions for data at all light levels.

A quantitative comparison to reconstructions with other least-squares algorithms is quite difficult because different authors[8,9] used different window functions for the spectrum. Even with the incorporation of higher spatial frequencies inherent in using the telescope MTF as a window function, the quality of our reconstructions is equivalent to the other methods.

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