BISPECTRAL PARAMETER ESTIMATION USING LEAST-SQUARES

A. Glindemann, R.G. Lane and J.C. Dainty

Blackett Laboratory, Imperial College, London SW7 2BZ, UK

Abstract

We present a least-squares algorithm for the phase reconstruction of two-dimensional images from the bispectrum phase. The method is successfully applied to experimental photon-limited data in the visible and to an extended infra red object (IRC +10216).

Additionally a new least-squares technique for model fitting the parameters of a binary star to the bispectral phase is introduced and applied to photon-limited data. Unlike earlier methods this algorithm does not require the measurement of a reference star as the bispectral phase is unaffected by the turbulent atmosphere and minor telescope aberrations.

1. INTRODUCTION

With the introduction of the bispectrum in speckle interferometry[1], the reconstruction of image intensities rather than their autocorrelations became possible by recovering the object phase from the bispectrum phase. Originally a recursive method was used to reconstruct the object phase[2,3]. The disadvantage of this method lies in the fact that the error in the bispectral phase for low spatial frequencies affects the recovered object phases at higher frequencies. The application of a least-squares method should give an advantage in this respect. A number of least-squares algorithms have been proposed recently[4–7] of which we adopted Haniff's method[6] for one-dimensional, simulated data. We have extended it to the two-dimensional case and applied it to experimental data[8].

Apart from this general approach to recover the object phase one can also restrict the possible object phase distributions by introducing constraints for the object to be reconstructed such as positivity and/or a certain shape (e.g. a binary star). The problem is then to find the unknown parameters of this object. Hofmann and Weigelt[9] have introduced a building block method where the number and the position of stars making up the image is optimized by applying a least-squares method to the bispectrum.

We present a slightly different approach by fitting only to the bispectral phase, an object phase that is parametrized by the three parameters of a binary, i.e. relative brightness, separation and position angle. Thus, unlike earlier methods, the measurement of a reference star

to compensate the speckle transfer function is not necessary[10]. An extension of this method to triple or multiple stars is possible.

2. BISPECTRUM

We consider two two-dimensional spaces called image and Fourier space in which arbitrary points are identified by the position vectors $\vec{x} = (x, y)$ and $\vec{u} = (u, v)$. An image $f(\vec{x})$ and its spectrum $F(\vec{u})$ constitute a Fourier transform pair, with $|F(\vec{u})|$ the modulus and $\phi(\vec{u})$ the phase of the spectrum.

The bispectrum of the real function $f(\vec{x})$ is defined as

$$\begin{aligned} F^{(3)}(\vec{u}_1, \vec{u}_2) &= F(\vec{u}_1)F(\vec{u}_2)F(-\vec{u}_1 - \vec{u}_2) \\ &= |F^{(3)}(\vec{u}_1, \vec{u}_2)|e^{i\psi(\vec{u}_1, \vec{u}_2)}, \end{aligned} \tag{1}$$

where $\psi(\vec{u}_1, \vec{u}_2)$ is the phase of the object bispectrum. The bispectrum is averaged over a sequence of short exposure speckle images to reduce the effects of the turbulent atmosphere. As shown by Lohmann et al[2] the phase of the averaged bispectrum is independent of minor telescope aberrations and the turbulent atmosphere. Thus

$$\psi(\vec{u}_1, \vec{u}_2) = \phi(\vec{u}_1) + \phi(\vec{u}_2) - \phi(\vec{u}_1 + \vec{u}_2), \tag{2}$$

with $\phi(\vec{u})$ the true object phase.

3. OBJECT PHASE RECONSTRUCTION

In practice the bispectrum phase is measured from a finite number of speckle frames. We denote this measurement of the bispectrum phase by $\tilde{\psi}$. The problem of reconstructing the object phase is simply a problem of determining which set of object phases is most consistent with the measured bispectrum phase.

Given an initial estimate of the object phase $\hat{\phi}$ it is possible to estimate the difference between the phase at a point in the measured bispectrum and that calculated from our current estimate of the object phase $\hat{\phi}$. This difference is given by

$$\Delta_{\vec{\imath},\vec{\jmath}} = \tilde{\psi}_{\vec{\imath},\vec{\jmath}} - (\hat{\phi}_{\vec{\imath}} + \hat{\phi}_{\vec{\jmath}} - \hat{\phi}_{\vec{\imath}+\vec{\jmath}}), \tag{3}$$

where $\vec{\imath}$ and $\vec{\jmath}$ are the discrete coordinates in the two-dimensional phase array. Solving (3) for all bispectral points by a least-squares method is not completely straightforward because the average bispectrum phase is calculated only over the interval ($-\pi$ to $+\pi$), whilst the bispectrum phase

obtained from the reconstructed object phase lies within (-3π, +3π). A classical least-squares method, when used to solve the above overdetermined system of equations, fails to incorporate the 2π phase ambiguity.

The procedure proposed by Haniff[6] is to minimise the weighted sum

$$\sum_{1}^{M} \mathrm{W}_{\vec{\imath},\vec{\jmath}} \ (\mathrm{mod}_{2\pi}(\tilde{\psi}_{\vec{\imath},\vec{\jmath}} - (\hat{\phi}_{\vec{\imath}} + \hat{\phi}_{\vec{\jmath}} - \hat{\phi}_{\vec{\imath}+\vec{\jmath}})))^2, \tag{4}$$

where the summation is over M selected points in the bispectrum and the use of the modulo 2π function incorporates the 2π phase ambiguity. $\mathrm{W}_{\vec{\imath},\vec{\jmath}}$ is a weighting that is assigned to each equation since not all bispectral phases are measured to the same accuracy. It is defined as the reciprocal of the variance of the bispectrum phase $\tilde{\psi}$ and is

$$\begin{aligned} \mathrm{W}_{\vec{\imath},\vec{\jmath}} &= \left[Var(\tilde{\psi}_{\vec{\imath},\vec{\jmath}})\right]^{-1} \\ &= \left[\frac{Var_{\perp}(F^{(3)}_{\vec{\imath},\vec{\jmath}})}{|F^{(3)}_{\vec{\imath},\vec{\jmath}}|^2}\right]^{-1}, \end{aligned} \tag{5}$$

with $Var_{\perp}(F^{(3)}_{\vec{\imath},\vec{\jmath}})$ the variance in the direction perpendicular to the direction of the complex bispectrum phasor $F^{(3)}_{\vec{\imath},\vec{\jmath}}$ and $|F^{(3)}_{\vec{\imath},\vec{\jmath}}|^2$ the square of the bispectral modulus.

In practice it is not computationally feasible to use the entire bispectrum of a two-dimensional array, and only those portions with a large signal to noise ratio (SNR) are employed. We define a subplane to be the set of all points in the bispectrum obtained by fixing $\vec{\imath}$ at a constant value and varying $\vec{\jmath}$. It has been noted by Ayers et al[3] that those subplanes for which $|\vec{\imath}|^2$ is small usually have a significantly higher SNR. Thus, when reconstructions are quoted as being for a given number of subplanes, we use those subplanes for which $|\vec{\imath}|^2$ is the smallest.

4. OBJECT MODULUS RECONSTRUCTION - THE WINDOW FUNCTION

The modulus of the object spectrum $|F(\vec{u})|$ usually is recovered after taking the square root of the quotient of the averaged power spectrum of the object, $< |\tilde{F}(\vec{u})|^2 >$, over the averaged power spectrum of the reference (point) star, $< |\tilde{F}_{ref}(\vec{u})|^2 >$. This procedure removes the telescope MTF and amplifies the noise in spatial frequencies near and beyond the diffraction limit. It is thus necessary to choose some form of window function, for example the telescope MTF, to prevent this noise dominating the reconstruction. Thus

$$|F(\vec{u})| = \mathrm{MTF}(\vec{u})\sqrt{\frac{< |\tilde{F}(\vec{u})|^2 >}{< |\tilde{F}_{ref}(\vec{u})|^2 >}}. \tag{6}$$

However, the influence of the window function is more severe than only producing more or less smooth reconstructions. Observing double stars, one is interested in their position and the

relative brightness. As the fringe pattern of the object spectrum is not reconstructed perfectly up to the diffraction limit but becomes noisier for high frequencies a wider window function eventually adds only noise to the spectrum. Fourier transforming the resulting spectrum yields a higher peak at the centre than using a narrower window function that includes fewer noisy values of the spectrum. Thus, the relative brightness depends on the width of the window function. For instance, we found for a particular set of data that the relative brightness of the fainter star of β Del is .31 if the telescope MTF is used and .37 if an MTF with a cutoff frequency at 65% of the diffraction limit is used (see Sect.6 for details about the data). The problem is that the window function cannot be defined properly because the noise increases continously.

So far, different authors used different window functions ranging from truncating the spectrum at 60% of the diffraction limit[7] to using a telescope aperture function, probably the MTF, with a cutoff frequency at 65% of the diffraction limit[5].

The use of the full telescope MTF appears to be the most logical choice of window function. This would allow easy comparison of the results from different reconstructions.

5. MODEL FITTING THE BISPECTRAL PHASE

For the special case of a binary it is possible to describe the object by

$$b(\vec{x}) = \delta(\vec{x}) + A\delta(\vec{x} - \vec{p}), \tag{7}$$

where the first star of brightness 1 is located at $\vec{x} = 0$ and the second one of brightness A at $\vec{x} = \vec{p}$. Thus the spectrum of the binary is given by

$$B(\vec{u}) = 1 + Ae^{i2\pi\vec{u}\vec{p}}, \tag{8}$$

and its bispectrum by

$$\begin{aligned} B^{(3)}(\vec{u_1}, \vec{u_2}) &= B(\vec{u_1})B(\vec{u_2})B(\vec{u_1} - \vec{u_2}) \\ &= 1 - A - A^2 + A^3 + (A + A^2)4\cos(\pi\vec{u_1}\vec{p})\cos(\pi\vec{u_2}\vec{p})\cos(\pi(\vec{u_1} + \vec{u_2})\vec{p}) + \\ &\quad i4(A^2 - A)\sin(\pi\vec{u_1}\vec{p})\sin(\pi\vec{u_2}\vec{p})\sin(\pi(\vec{u_1} + \vec{u_2})\vec{p}). \end{aligned} \tag{9}$$

The procedure advocated in this paper is to select $\vec{p}$ and A to minimise, in a weighted least squares sense, the difference between the bispectrum phase computed from the observations, $\tilde{\psi}_{\vec{i},\vec{j}}$ and that obtained by (9), $\beta_{\vec{i},\vec{j}}$.

We thus minimise

$$\sum_{1}^{M}(\mathrm{mod}_{2\pi}(\tilde{\psi}_{\vec{i},\vec{j}} - \beta_{\vec{i},\vec{j}}))^2 W_{\vec{i},\vec{j}}, \tag{10}$$

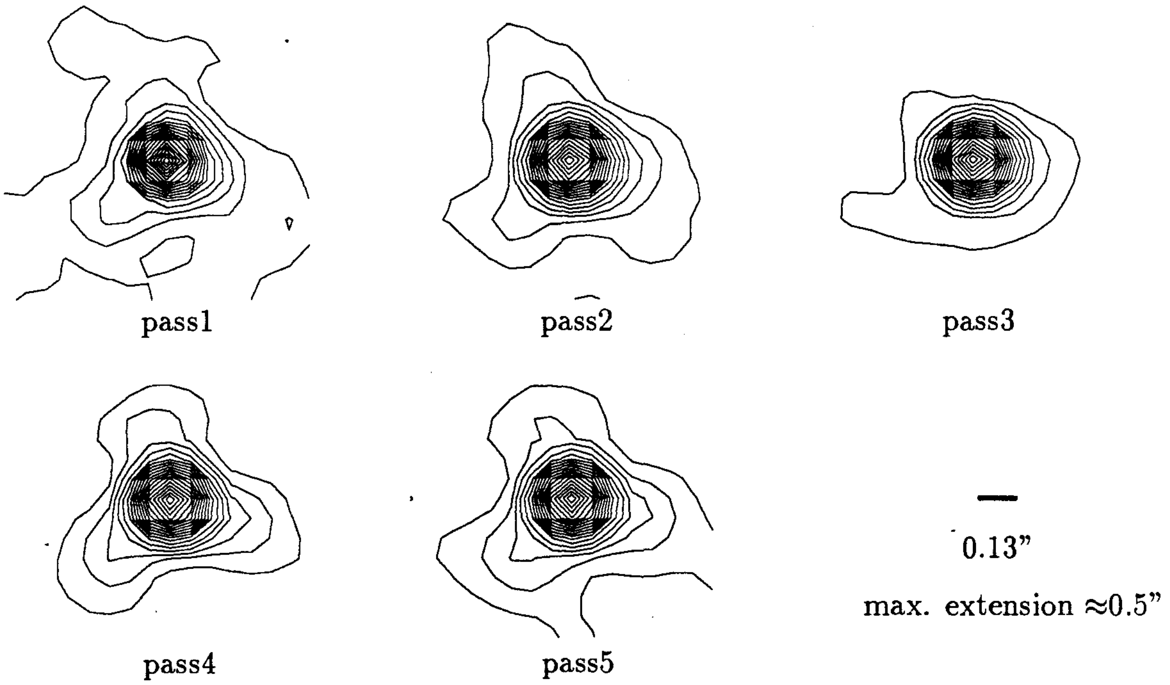


Figure 1: Reconstruction of IRC +10216 for five different passes. 80 subplanes are used for the phase reconstruction and 20 contours are displayed. The diffraction limit of the telescope is 0.13".

over all five passes[16,17]. The variation between different passes is greater than between different reconstruction methods. Possibly the change of statistical parameters of the atmosphere between passes for the object and reference star is responsible for this variation. A similar variety in the reconstruction occured when phase retrieval on the power spectrum was applied. It has been suggested that the triangular shape of the reconstruction could in part be caused by aberrations of the telescope. However, this structure was not apparent in our reconstructions of the reference star and in principle the bispectrum technique should be imune to small telescope aberrations.

– Binary data

The first binary data set consists of high light level images observed on the San Martir Observatory 2.12 m telescope, UNAM, Mexico, at $\lambda = 516$ nm in October 1988. This data has been provided by J. Ohtsubo who also presented a power spectrum analysis[13,14]. The second data set, provided by E. K. Hege, is photon limited and was taken on the Steward Observatory 2.3 m telescope, University of Arizona, at $\lambda = 550$ nm in October 1986 using the Stanford University MAMA detector[11]; this data has been reconstructed previously by Meng et al[5], Pérez-Ilzarbe et al[15] and Glindemann et al[8].

where $\beta_{\vec{i},\vec{j}}$ is the phase computed from the current estimates of the binary star parameters and the difference is taken modulo 2π. The weighting $W_{\vec{i},\vec{j}}$ is again chosen to be the variance of the bispectrum phase $\tilde{\psi}$.

6. EXPERIMENTAL RESULTS

The least-squares routine E04DGF from the NAG library[18] was used to minimise the cost function (4) for the general phase reconstruction. The initial estimate for the phase can be zero or a random number between $\pm\pi/2$, producing always the same results. A random number of a wider interval introduces a large linear term in the phase reconstruction shifting the reconstructed image to the border of the array and decreasing the image quality a little bit.

For the binary model fit, the cost function defined by (10) has multiple minima and the application of the minimisation algorithm did not yield consistent results for the parameters when started from an arbitrary initial estimate of the binary star parameters. Fortunately, it is relatively easy to obtain a crude estimate of the relative position of the binary stars from the uncorrected power spectrum of the speckle images. This yields two possible positions on opposite sides of the star at the center. After using both for the least squares minimization with the E04DGF or E04HFF routine, one of them was always identifiable as the "true" position because it had a smaller error sum. The initial estimate of the relative brightness A is not critical and was set to either 0.5 or 0.9 without significant differences in the final solution.

The maximal number of subplanes used for our reconstructions is 80, which is 1/20 of all subplanes in a two-dimensional array with a diameter of 64 pixels. The highest spatial frequency of the fixed frequency vector of a subplane $|\vec{i}|$ for this case is about 1/4 of the diffraction limit of the telescope. In most cases this frequency is beyond the limit given by the Fried parameter, so that bispectral values with a low SNR are also taken.

– Infra red data

This data of an extended object (IRC +10216, courtesy of J. Christou and C. Haniff) was taken at $\lambda = 2.2\ \mu m$ on a 4m telescope (diffraction limit of resolution 0.13") with the NOAO 2-D IR speckle camera in May 1989. Five passes of approximately 200 frames each were recorded together with three passes for the reference star 31 Leo. The array size is 64x64 and the bispectrum phase was taken from a 64x64 array in the object spectrum.

Fig. 1 shows the reconstructed image intensities for five different passes using 80 subplanes. There is no significant difference compared to other reconstructions on the same data averaging

Table 1 shows the model fit of the binary parameters relative brightness A, separation and orientation for μ-Ori, using only a subset of 2000 frames for each model fit. Thus it is possible to get a crude estimate for the accuracy. Applying our method to the frames 2100 to 4100 did not give consistent results for any number of subplanes; the orientation was by 20 to 60° out of the expected value. However, the investigation of the data, using semi-long exposures of 250 frames, revealed that the data for these frame numbers was apodized due to an obstacle in the optical path or a tracking error. Thus it is not surprising that the parameters could not be determined.

A rough estimation of the accuracy of the parameters gives ±0.02 for the relative brightness, ±0.005" for the separation and ±1° for the orientation. A similar consistency was determined when using only every second and every fourth photo event of the 126-Tau data[10]. It should be emphasized that these numbers can only give an impression of the consistency of the method. Only after a long period of application to more experimental data and comparison to existing methods a proper accuracy measure can be determined. However, it is very difficult to find an unbiased estimate of the relative brightness of the two stars with existing methods.

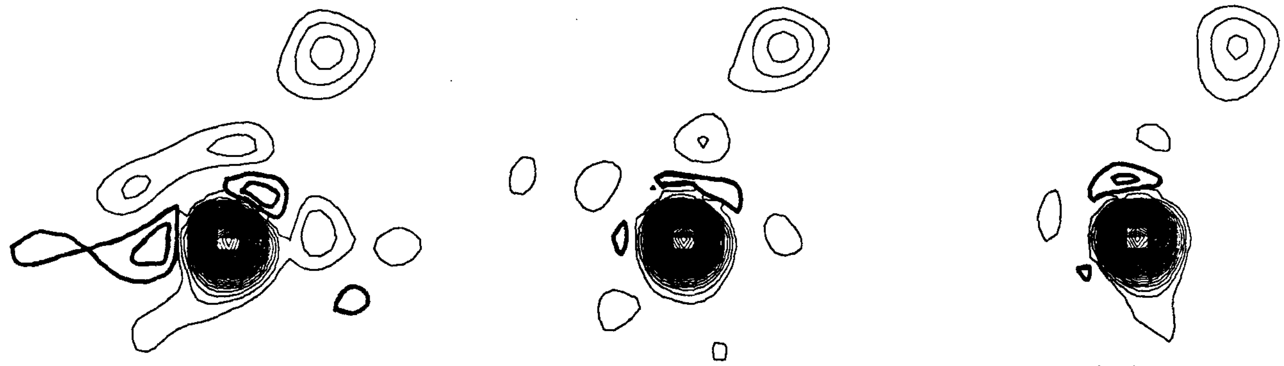
Star name	No. of Events	No. of frames	Rel. Br.	Sep.["]	Orient.[°]	McAlister[12]: Sep.["]	Orient.[°]
β-Del	685	4500	0.37	0.184	101.6	0.172 – 0.20	85 – 136.0
μ-Ori	433	7100	0.16	0.219	26.3	0.218	206.4
126-Tau	610	4900	0.26	0.353	240.4	(not available)	
ADS4299	244	12850	0.31	0.121	138.1	0.135	125.2

Table 2: Results for photon limited binary data. The numbers are the average over reconstructions with 44, 60 and 80 subplanes of the bispectrum.

Table 2 shows the three parameters for β-Del, μ-Ori, 126-Tau and ADS4299. The two observations of β-Del quoted in the McAlister catalogue were done before and after the measurements we use, and the observation of ADS4299 quoted in the catalogue was performed before this data was taken. The results for the high light level data are presented in table 3. All results were obtained using the average of model fits with 44, 60 and 80 subplanes of the bispectrum.

7. CONCLUSIONS

The least-squares algorithm for the phase reconstruction of two dimensional images proved to be robust and reliable for the general reconstruction of the phase as well as for model fitting to



10 subplanes 28 subplanes 60 subplanes

Figure 2: Reconstructions of μ-Ori for different numbers of subplanes. 7100 frames with an average of 433 events per frame are used. Thick lines indicate negative values, 33 contours are displayed. The window function is the full telescope MTF. For the reconstruction with 10 subplanes, the CPU time is 6 minutes on a SUN Sparc 1 and the number of iterations 108. For 28 subplanes 13 minutes and 91 iterations and for 60 subplanes 23 minutes and 80 iterations.

The photon limited data was registered on a 256x256 array and the high light level data on a 128x128 array. The bispectrum was taken from a 64x64 (32x32 for the high light level data) object phase array in Fourier space. Thus, 80 subplanes means 5% (20% for the high light level data) of the whole, non-redundant bispectrum.

Image intensity reconstructions of μ-Ori for different numbers of subplanes are displayed in Figure 2. It is readily apparent that the image quality increases with increasing number of subplanes.

No. of	frames 1–2000			frames 4200–6200		
Subpl.	Rel. Br.	Sep.["]	Orient.[°]	Rel. Br.	Sep.["]	Orient.[°]
6	0.09	0.209	28.5	0.16	0.221	27.1
10	0.13	0.211	27.6	0.17	0.220	27.7
18	0.16	0.215	27.2	0.18	0.222	27.3
28	0.16	0.219	28.0	0.18	0.222	27.1
44	0.16	0.221	28.7	0.18	0.222	27.0
60	0.16	0.223	28.8	0.19	0.220	26.3
80	0.15	0.226	27.8	0.19	0.220	26.3

Table 1: Results for μ-Ori with 433 events per frame. The parameters from the McAlister catalogue[12] for the same period of time are 0.218" separation and 206.4° (± 180°) orientation.

Star name	Combined Magnitude	No. of frames	Rel.Br.	Sep.["]	Orient.[°]	Isobe[14]: Sep.["]	Orient.[°]
ADS2253	6.7	64	0.53	0.644	83.5	0.50	265
ADS2980	7.4	74	0.06	0.770	148.5	0.60	325
ADS3390	7.8	74	0.25	1.243	14.2	1.2	15
ADS5871	6.4	74	0.73	1.607	319.7	1.20	320
ADS15267	7.4	74	0.65	0.417	253.6	0.30	75
ADS15281	4.1	74	0.76	0.263	101.7	0.22	96
ADS15992	8.0	64	0.16	0.595	49.9	0.50	52
ADS16836	5.0	74	0.55	0.675	87.1	0.50	89

Table 3: Results for high light level binary data. The numbers are the average over reconstructions with 44, 60 and 80 subplanes of the bispectrum.

the bispectral phase the binary star parameters separation, orientation and relative brightness, the latter parameter being particularly difficult to come by with other techniques. Unlike earlier methods the model fitting algorithm does not require the measurement of a reference star as the bispectral phase is unaffected by the turbulent atmosphere and minor telescope aberrations.

We observe that the quality and stability of the reconstruction and the model fit improves with increasing the number of subplanes, both for the general reconstruction and for the model fit. So the higher computational burden (typically 30 minutes on a SUN-Sparc 1+ workstation for 80 subplanes compared to 1 minute for 6 subplanes) seems to be justified.

ACKNOWLEDGEMENTS

We would like to thank J. Christou, C. Haniff, E.K. Hege and J. Ohtsubo for providing the astronomical data. Financial support by a SERC grant (GR/F 75544) and by a grant of the Deutsche Forschungsgemeinschaft for one of us (A.G.) is gratefully acknowlegded.

REFERENCES

1. G. Weigelt, Opt. Commun., **21**, 55–59, (1977).
2. A.W. Lohmann, G. Weigelt and B. Wirnitzer, Appl. Opt., **22**, 4028–4037, (1983).
3. G.R. Ayers, M.J. Northcott and J.C. Dainty, J. Opt. Soc Am. A., **5**, pp 963–985, (1988).

4. J.C. Marron, P.P. Sanchez and J.C. Sullivan, J.Opt. Soc. Am. A., **7**, 14–20, (1990).

5. J. Meng, G.J.M. Aitken, E.K. Hege and J.S. Morgan, J.Opt. Soc. Am. A., **7**, 1243–1250, (1990).

6. C.A. Haniff, J. Opt. Soc. Am. A, **8**, pp 134–140, (1991).

7. C.L. Matson, to be published in J. Opt. Soc. Am. A.

8. A. Glindemann, R.G. Lane and J.C. Dainty, in *Digital Signal Processing – 91*, V. Cappellini and A.G. Constantinides (eds), Elsevier, Amsterdam, 59–65, (1991).

9. K.-H. Hofmann and G. Weigelt, SPIE Proc. **1351**, pp 522–525, (1990).

10. A. Glindemann, R.G. Lane and J.C. Dainty, to be published in J. Opt. Soc. Am. A.

11. J.G. Timothy and J.S. Morgan, SPIE Proc. **627**, pp 654–659, (1986).

12. H.A. McAlister and W.I. Hartkopf, *"Second catalog of interferometric measurements of binary stars"*, Center for High Angular Resolution Astronomy, Contribution No. 2, October 1988.

13. S. Isobe, Y. Norimoto, M. Noguchi, J. Ohtsubo, N. Baba, N. Miura, H. Yanaka and T. Tanaka, Publ. Natl. Astron. Obs. Japan, **1**, pp 217–225, (1990).

14. S. Isobe, Y. Norimoto, M. Noguchi, J. Ohtsubo, N. Baba, N. Miura, H. Yanaka and T. Tanaka, Publ. Natl. Astron. Obs. Japan, **1**, pp 381–392, (1990).

15. M.J.Pérez-Ilzarbe and M. Nieto-Vesperinas, J. Opt. Soc. Am. A., **8**, pp 908–918, (1991).

16. J. Christou, SPIE Proc. **1237**, pp 424–435, (1990).

17. C.A. Haniff, D.A. Buscher, J.C. Christou and S.T. Ridgway, SPIE Proc. **1237**, pp 259–271, (1990).

18. The NAG library is available from NAG Inc, 1400 Opus Place, Suite 200, Downers Grove, IL 60515-5702, USA.