Space–time photon correlation using a 2-D photon event detector

D. Newman, A. A. D. Canas, and J. C. Dainty

A 2-D photon event detector, consisting of a microchannel plate coupled to a resistive anode for position readout, has been used to measure the spatiotemporal correlation of dynamic speckle. In this paper we discuss the implementation of this detector for these measurements and the effect of detector dark count, dead time, spatiotemporal integration, and other inherent biases. Initial results on dynamic speckle produced by a rotating diffuser are presented, and these agree well with theoretical predictions.

I. Introduction

The correlation properties of scattered radiation is an area of extensive research. At optical frequencies, perhaps the most productive experimental technique for these measurements involves photon counting with photomultipliers at low optical field intensities. Purpose built photon correlators are capable of measuring the temporal correlation properties of intensity fluctuations on time scales \( \tau > 10^{-7} \) sec in an autocorrelation (one detector) or cross-correlation (two-detector) mode with excellent signal-to-noise characteristics. For a large number of applications, however, it is quite useful to measure both the spatial and temporal intensity correlation properties simultaneously. For Gaussian processes this gives a complete dynamic picture of the stochastic intensity fluctuations and could be used in a number of applications including atmospheric or fluid flow rate measurements and real time stellar speckle interferometry.

In this paper we present preliminary results obtained using a space–time correlation system. The detector was manufactured by Instrument Technology, Ltd. (ITL) and consists of an 18-mm diam photocathode and a position sensitive anode capable of detecting photon events with a resolution spot size of \( \approx 90 \) \( \mu \)m. Although the image area can be addressed over a 1024 \( \times \) 1024-pixel area, we use the most significant 8 bits of the address to give 256 \( \times \) 256 pixels of dimensions 70 \( \times \) 70 \( \mu \)m. Each photoevent is processed as a 16-bit \( x,y \) coordinate and is subsequently time tagged (16 bits) and stored in computer memory. After the data are collected (typically \( 5 \times 10^4 \) photons), a differencing algorithm is used to create a histogram of space–time photoevent coordinate differences. This difference histogram is in fact the measured correlation function as is shown in Sec. III.

To demonstrate the versatility of this arrangement, we measured the space–time correlation properties of dynamic speckle from a rotating ground glass diffuser. As is well known the dynamic speckle generated by a coherent light source illuminating a moving diffuser contains detailed information about the deterministic motion of the diffuser (speckle velocimetry) and information concerning the size and shape of the source via the correlation area and correlation time of the fluctuating intensity pattern. The use of an imaging photon detector (IPD) allows one to study the evolution of the pattern either at a series of fixed points or in the rest frame of the moving speckle pattern. The latter option is extremely useful for measuring the purely stochastic spatial and temporal properties, whereas the former is used to determine the magnitude and direction of the deterministic motion.

In Sec. II a theoretical analysis of the space–time correlation function of speckle from a rotating diffuser is given paying close attention to the nature of the well-known bodily rotation of the pattern in the far field and its stochastic time evolution. In Sec. III we describe in some detail the actual implementation of the IPD for space–time correlation measurements. An analysis of the SNR of these measurements and the effect of dark count, dead time, spatiotemporal integration, and various other inherent biases on the measured correlation function are presented. In Sec. IV we show results of two experiments involving a moving speckle patterns and compare them to the theory of Sec. II.
II. Dynamic Speckle from a Rotating Diffuser

The scattering geometry is shown in Fig. 1. A Gaussian He–Ne laser beam is scattered by a pure phase diffuser which rotates in the \((\xi, \eta)\) plane with constant angular speed \(\omega\). In the Fresnel zone of the scattering plane, the speckle pattern appears to rotate, bodily, continually changing form as it moves. In other words, the pattern is rotating and boiling simultaneously as may be easily demonstrated in the laboratory. In the following analysis we obtain an expression for the spatiotemporal correlation of intensity fluctuations of the far-field dynamic speckle pattern. The transmission geometry of Fig. 1 is similar to that of Crosignani\(^4\) who has analyzed the temporal correlation structure of this process. More recently, Churnside\(^5\) has carefully analyzed this problem in a reflection geometry showing both the rotation and scintillation structure of the speckle pattern in the Fresnel zone. In our analysis we assume a perfectly plane wave source using the transmission geometry (Fig. 1). The final expression makes use of a simplifying approximation and is consistent with Churnside’s more general result.

Let \((\xi, \eta, 0)\) and \((x, y, z)\) define two parallel planes as shown in Fig. 1. The diffuser rotates in the \((\xi, \eta)\) plane about the origin of coordinates, and a Gaussian beam centered at \((0, R, 0)\) illuminates the rotating diffuser. The unperturbed beam has an amplitude profile

\[ U_0(\xi, \eta) = A_0 \exp\left(-[(\xi^2 + (\eta - R)^2)]/W^2\right), \]  

where \(W\) is the \(1/e^2\) beam intensity radius, and \(A_0\) refers to the complex plane wave amplitude at \(z = 0\). The transmitted amplitude immediately after the diffuser is then

\[ U(\xi, \eta, t) = U_0(\xi, \eta) \exp[i\phi(\xi, \eta, t)]. \]  

where \(\phi(\cdot)\) is the random phase term and has time dependence since the diffuser is moving. The complex amplitude at a point \((x, y)\) in the Fresnel region is given by

\[ U(x, y, t) = K \int_{-\infty}^{\infty} U(\xi, \eta, t) \exp\left[i\frac{\pi}{\lambda z} \left((\xi - x)^2 + (\eta - y)^2\right)\right] d\xi d\eta, \]  

where \(K\) is an unimportant scaling factor and will be ignored in the following. Defining the Fresnel transfer function \(f\) as

\[ f(\xi, \eta; x, y) = U_0(\xi, \eta) \exp\left[i\frac{\pi}{\lambda z} \left(x^2 + y^2 - 2(x\xi + y\eta)\right)\right], \]  

we have

\[ U(x, y, t) = \int_{-\infty}^{\infty} f(\xi, \eta; x, y) \exp[i\phi(\xi, \eta, t)] d\xi d\eta. \]  

The correlation of complex amplitudes is defined by

\[ \Gamma(\Delta x, \Delta y, r) = \langle U(x, y, t) U^*(x + \Delta x, y + \Delta y, t + r) \rangle, \]  

where \(\langle \cdot \rangle\) refers to an ensemble average.

For a Gaussian random process, the correlation of intensity fluctuations is simply

\[ C_I(\Delta x, \Delta y, r) = 1 + C_{II}(\Delta x, \Delta y, r), \]  

where

\[ C_{II}(\Delta x, \Delta y, r) = \frac{[\Gamma(\Delta x, \Delta y, r)]^2}{[\Gamma(0, 0, 0)]^\Delta}, \]  

shown here in normalized form so that \(0 \leq |C_{II}(\Delta x, \Delta y, r)| \leq 1\). From Eqs. (4), (5) and (6) we have
The spatiotemporal correlation of intensity fluctuations is the product of three Gaussians and has a straightforward interpretation. The first term gives the overall temporal decorrelation envelope of \(1/\tau_0\) time \(\tau_0 = W/R_w\). This is simply the time taken for the diffuser to move through an arc of length \(W\), the beamwidth. The remaining terms describe the translation of the speckle pattern as a function of the absolute positions \(x_0, y_0\) and time \(t\). The correlation function along the \(\Delta x, \Delta y\) directions is peaked whenever
\[
\Delta x = -y_0 \omega t - y_0 \omega t,
\]
\[
\Delta y = x_0 \omega t - x_0 \omega t,
\]
where \((x_0, y_0)\) is the mean absolute position with respect to the beam axis in the far field. A little thought shows that the translational motion of the speckle pattern for small \(\theta\) is along an arc of length \(\Delta s = (\Delta x^2 + \Delta y^2)^{1/2}\) traversed by rotating the vector \(\rho = (x_0, y_0)\) through an angle \(\omega t\). Thus the speckle pattern appears to rotate bodily in the far field about the point \((0,0)\) in the far field, its translational velocity linearly increasing with the radial vector \(\rho = (x_0, y_0)\). The spatial width of the correlation function is given by the parameter \(r = \lambda z/\pi W\) hence for fixed time delay \(t\) the spatial correlation function is a Gaussian of \(1/\tau_0\) width \(r\) and has a peak value determined by \(\exp(-R_0^2/2W^2)\) centered on the difference coordinates \(\Delta x = -y_0 \omega t, \Delta y = x_0 \omega t\). The speed and direction of this motion may be determined from observing the coordinates of the spatial correlation peak at various time lags assuming \(\tau \leq \tau_c\). The velocity vector is given by
\[
\nu_{\omega t} = (\Delta r_2 - \Delta r_1)/(\tau_2 - \tau_1),
\]
where \(\Delta r_1 = (\Delta x_1, \Delta y_1)\) and refers to the location of the spatial correlation peak at time interval \(\tau_1\) and similarly for \(\Delta r_2\).

One last point of interest is the dependence of the correlation function on the parameter \(R\). Whenever \(R/W \gg x_0/r\) and \(y_0/r\) the decorrelation at \((x_0, y_0)\) is a much stronger effect than the translation. Thus for detection locations \(|\rho|/r \ll R/W\), i.e., at points near the center of the diffraction field the speckle tends to boil more than rotate. As \(R\) is increased the region where the boiling effect dominates will increase. Conversely, in the limit \(R \to 0\) the speckle pattern simply rotates without decorrelating as in this case the optical axis coincides with the diffusers’s center of rotation; hence the beam illuminates the same set of scatterers which simply move in a circular path.

III. Experimental Design

In this section we discuss various aspects of measuring the normalized correlation function \(C_f(\Delta x, \Delta y, t)\) experimentally using the imaging photon detector (IPD) and time-marking electronics. The proximity focused IPD is based on an S20 photocathode, a triple microchannel plate (MCP) intensifier, and a resistive anode readout system. Details of the detector development and performance have been described elsewhere\(^6,7\) and will not be repeated here. The following
showing it to be a very efficient, but biased, estimator of $C_f(\Delta x, \Delta y, \tau)$ in general. Section III.B describes the effect of dark count, dead time, spatial, and spatiotemporal integration on the measured correlation function. In Sec. III.C we give an estimate of the SNR characteristics of the measured correlation function for this type of implementation.

### A. Photon Differencing Algorithm

For an ideal detector the cross-correlation of the intensities at the points $p = (x,y)$ and $p' = (x + \Delta x, y + \Delta y)$ may be expressed as

$$C(p' - p, \tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} I(t) I(t + \tau) dt,$$  \hspace{1cm} (13)

where $I(t)$, $I'(t)$ are the time-dependent intensity signals at $p, p'$, respectively, and we assume $I(t)$ is a spatially and temporally stationary random process so that the correlation function depends only on spatial and temporal coordinate differences.

Detectors such as photomultipliers encode the intensity signal as a series of discrete pulses where the pulse rate varies in proportion to the intensity level. If we select a fixed integration period $\Delta t$, a sampling time at such that $\Delta t \ll \tau$, the fluctuating intensity pattern is approximately frozen for the duration of the integration time.

Defining $U$ as the expected number of photoelectron pulses measured during the sampling interval, we have

$$U = \alpha \Delta t I(t), \hspace{1cm} (14)$$

where $I(t)$ is the slowly varying classical intensity ($\Delta t/\tau_c \ll 1$) and $\alpha$ is the quantum efficiency of the detector. Equation (14) gives the expected value of the measured photon rate for a single realization of the intensity. As the intensity fluctuates in time, the measured photon rate also fluctuates with a probability density function $P(U) = \alpha \Delta t I(t)$, where

$$P(I) = \frac{1}{I} \exp(-I/I) \hspace{1cm} (15)$$

is the well-known probability density of intensity for Gaussian speckle.

The probability of measuring $n$ photons during the time interval $\Delta t$ is given by the compound or doubly stochastic Poisson distribution

$$P(n) = \int_0^\infty P(U) \frac{I^n}{n!} \exp(-U) dU. \hspace{1cm} (16)$$

The average number of detected photons per sampling time is then

$$\langle n \rangle = \sum_{n=1}^\infty n P(n) = \langle U \rangle \hspace{1cm} (17)$$

Furthermore, $\langle U \rangle = \alpha \Delta t \langle I \rangle$ from Eq. (14) so that $\langle n \rangle$ is directly proportional to the mean intensity.

The joint probability density of measuring $n$ photoelectrons at the space–time point $(x,y,t)$ and $m$ photoelectrons at $(x + \Delta x,y + \Delta y,t + \tau)$ is a straightforward generalization of Eq. (16):

$$P(n,m) = \int_0^\infty P(U_1,U_2) \frac{U_1^n U_2^m}{n! m!} dU_1 dU_2 \hspace{1cm} (18)$$

$$= \frac{1}{n! m!} \exp[-(U_1 + U_2)][U_1^n U_2^m]. \hspace{1cm} (18)$$

This leads to the important result

$$\langle nm \rangle = \sum_{n,m=1}^\infty n P(n,m) = \langle U_1 U_2 \rangle = \langle \alpha \Delta t \rangle^2 \langle I_1 I_2 \rangle, \hspace{1cm} (19)$$

where $I_1 = I(x,y,t), I_2 = I(x + \Delta x, y + \Delta y,t + \tau)$ and $\tau = k \Delta t$, where $k \geq 0$ is an integer. Combining Eqs. (17) and (19) yields

$$\frac{\langle nm \rangle}{\langle n \rangle \langle m \rangle} = \frac{\langle I_1 I_2 \rangle}{\langle I_1 \rangle \langle I_2 \rangle} = 1 + C_d(\Delta x, \Delta y, \tau). \hspace{1cm} (20)$$

Thus the normalized correlation of photoelectron number fluctuations is identical to the normalized correlation of classical intensities defined in the previous section.

In a real experiment, the ensemble average (19) is estimated by taking a time average over a finite number $N$ of sample times and forming the sum

$$\langle nm \rangle \sim \frac{1}{N-k} \sum_{i=1}^{N-k} n_i m_{i+k}, \hspace{1cm} (21)$$

where $n_i$ refers to the number of photoevents detected at $(x,y)$ during the $i$th sample time, and $m_{i+k}$ is the number detected at $(x + \Delta x, y + \Delta y)$ during the $(i + k)$th sample time. Equation (21) is used in conventional photon correlators where $k \ll N$ and $N$ is usually very large ($>10^9$).

To normalize Eq. (21), we divide by $\langle n \rangle \langle m \rangle$ where

$$\langle n \rangle \sim \frac{1}{N-k} \sum_{i=1}^{N-k} n_i, \hspace{1cm} \langle m \rangle \sim \frac{1}{N-k} \sum_{i=1}^{N-k} m_i. \hspace{1cm} (22)$$

For a stationary process, $\langle n \rangle = \langle m \rangle$ at all points in the detection plane. In the limit of $N \to \infty$, Eqs. (21) and (22) have been shown to represent unbiased estimators of the ensemble averaged quantities $\langle nm \rangle$, $\langle n \rangle$, and $\langle m \rangle$.

In our experiment we measure the normalized correlation function $\langle nm \rangle$ at large numbers of spatial and temporal lags simultaneously. This is accomplished by a photon coordinate differencing algorithm which shall now be described. The IPD detects photon events over a $256 \times 256$-pixel area and serially encodes each data point as a 16-bit number (8-bit $x$, 8-bit $y$). The minimum time between valid events, i.e., the detector’s (global) dead time, is $\sim 1.5 \times 10^{-6}$ sec. An external interface contains a running clock of time $\Delta t$ which is also encoded as a 16-bit number, and the final 32-bit space-time coordinate is stored as an array variable in an HP 9826 computer. For purposes of the following discussion we assume that each valid photon event coordinate is stored in three separate arrays $X(I)$, $Y(I)$, $T(I)$ in the computer. The array variables are, therefore, a time ordered list of photon event coordinates. After a buffer of data has been filled ($\sim 5 \times 10^4$ photons) a histogram of space–time coordinate differences is created using the following type of subroutine:
The function \( H(\Delta x, \Delta y, r) \) may be interpreted as a coincidence histogram of all pairs of events separated by the space–time vector \((\Delta x, \Delta y, r)\). In our experiment typical values of the various parameters were \( \text{MAX} = 5 \times 10^4 \) events recorded over \( N = 10^5 \) sample times \( \Delta t \), \( r_{\text{max}} = 10 \Delta t \). Under these conditions the total number of differences generated is \( N^2 \times \tau_{\text{max}}/\Delta t = 10^7 \). A machine code subroutine processed each iteration in \( \sim 25 \) \( \mu \text{s} \) giving an average of 4 min of total computing time.

It remains to be seen how the differencing algorithm yields the correlation function at all space–time lags. This can be demonstrated by a simple example in which we consider only two pixels in the detector array \( \mathbf{p} = (x, y) \) and \( \mathbf{p}' = (x + \Delta x, y + \Delta y) \). An imaginary data list would look something like

<table>
<thead>
<tr>
<th>Array List</th>
<th>Time</th>
<th>Events/sample time</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x,y,1 )</td>
<td>1</td>
<td>( n_1 = 2, n_{1}' = 1 )</td>
</tr>
<tr>
<td>( x',y,1 )</td>
<td>1</td>
<td>( n_2 = 0, n_{2}' = 2 )</td>
</tr>
<tr>
<td>( x,y,2 )</td>
<td>2</td>
<td>( n_3 = 2, n_{3}' = 1 )</td>
</tr>
<tr>
<td>( x',y,2 )</td>
<td>2</td>
<td>( n_4 = 1, n_{4}' = 1 )</td>
</tr>
</tbody>
</table>

Applying the differencing algorithm will clearly give nonzero values for \( H(\Delta x, \Delta y, r) \), \( H(-\Delta x, \Delta y, r) \), and \( H(0,0,r) \), where \( 0 \leq r \leq 3\Delta t \). As an example it is easy to check that

\[
H(\Delta x, \Delta y, 0) = 2 = \frac{1}{2} \sum_{i=1}^{4} n_{i} n_{i}',
\]

\[
H(\Delta x, \Delta y, 1) = 6 = \sum_{i=1}^{3} n_{i} n_{i+1}',
\]

\[
H(-\Delta x, -\Delta y, 1) = 6 = \sum_{i=1}^{3} n_{i} n_{i+1}',
\]

\[
H(0,0,1) = 7 = \sum_{i=1}^{3} n_{i} n_{i+1} + \sum_{i=1}^{3} n_{i} n_{i+1}'.
\]

and so on. Thus for \( r \geq 1 \) the differencing algorithm gives the same type of information as Eq. (18), and is easily generalized for an arbitrary number of detectors with the same basic result. From Eq. (19) we see that each term

\[
\sum_{i=1}^{N} n_{i} n_{i+h}',
\]

is a good estimate of the correlation function at a particular space–time lag (ignoring the normalizing constant) whenever \( N \) is large enough to encompass many independent realizations of the fluctuating intensity pattern and \( k \ll N \). Note that there are two such terms for \( \Delta p = (0,0) \) and only one term for \( \Delta p = (\pm \Delta x, \pm \Delta y) \). This means that with two detectors at \( \mathbf{p}, \mathbf{p}' \) there are two independent ways of generating the vector difference \( \Delta \mathbf{p} = (0,0) \) and only one way to generate \( \Delta \mathbf{p} = (\pm \Delta x, \pm \Delta y) \) for a fixed-time lag \( r \). A linear array of \( N_x \) equally spaced detectors yields \( N_x \) independent ways to generate \( \Delta x = 0, N_x - 1 \) ways to generate \( \Delta x = \pm 1 \), and \( N_x - (N_x - 1) \) ways to generate \( \Delta x = \pm N_x \).

Since the statistics of the fluctuating intensity pattern are isotropic, this means that the measured correlation histogram for fixed time lag \( r \) is biased by a multiplicative factor \( G(\Delta x) = N_x^{-1} \left( N_x - |\Delta x| \right) \) due to the finite size of the linear array. In two dimensions the measured correlation histogram is related to the desired correlation function by

\[
C_x(\Delta x, \Delta y, r) = \frac{1}{N(n)^2} \left[ G(\Delta x) G(\Delta y) \right]^{-1} H(\Delta x, \Delta y, r) + \Theta(1/N),
\]

where

\[
G(\Delta x) G(\Delta y) = \left(1 - \frac{|\Delta x|}{N_x}ight) \left(1 - \frac{|\Delta y|}{N_y}\right).
\]

Equation (23) indicates that the differencing algorithm may be used to estimate the true normalized correlation function at all spatial and temporal lags of interest assuming the previously mentioned constraints are satisfied. It is also clear from this discussion that the SNR of the measured correlation function decreases with increasing spatial lag \(|\Delta x| \) \(|\Delta y| \) for fixed time lag \( r \). This will be discussed in Sec. III.C.

B. Systematic Biases

In principle, the normalized correlation function \( C_x(\Delta x, \Delta y, r) \) can be measured to any desired statistical accuracy under ideal experimental conditions. This section describes three very common nonideal systematic effects which generally alter the measured correlation function in an undesirable way. The three effects are due to dark count, dead time, and spatial integration. Each is shown to yield a manageable small deviation from the ideal case under appropriate experimental conditions.

The dark count is essentially uncorrelated noise due to thermionic emission in the photocathode. At fixed cathode voltages the average rate of this emission is reasonably constant over measurement times of a few seconds. If the number of dark count pulses in a particular sample time \( \Delta t \) is denoted \( n_d \), the number of pulses due to the true signal during this sample time is \( n \), the number of pulses measured in the presence of dark count is clearly \( n' = n + n_d \). Defining \( \delta = \langle n_d \rangle / \langle n \rangle \), where \( \langle \rangle \) denotes the ensemble average over the statistics of the fluctuating intensity, and assuming \( \delta \ll 1 \), the measured cross-correlation is given by

\[
C'_x(\Delta x, \Delta y, r) = C_x(\Delta x, \Delta y, r) + \Theta(1/N) \left(1 - G(\Delta x) G(\Delta y)\right) H(\Delta x, \Delta y, r).
\]
\[
\begin{align*}
C'_i &= \frac{\langle n' m' \rangle}{\langle n' \rangle \langle m' \rangle} = \frac{\langle n m \rangle}{\langle n \rangle^2} (1 - 2\delta) + 2\delta \langle \delta^2 \rangle,
\end{align*}
\]
where \(C'_i\) is the measured cross-correlation in the presence of the dark count and
\[
\frac{\langle nm \rangle}{\langle n \rangle^2} = 1 + C_{\Delta t}
\]
is the cross-correlation of the true signal. We again assume a stationary random process, and thus \(\langle n \rangle = \langle m \rangle\). The measured correlation of intensity fluctuations is then
\[
C_{\Delta t} = C_{\Delta t}(1 - 2\delta) + O(\delta^2).
\]
Therefore, the measured correlation function is decreased by a constant scaling factor \((1 - 2\delta)\) to first order in \(\delta\). For our experiment, \(\delta \sim 10^{-3}\) so this effect was extremely small.

Photon counting experiments with the IPD are also limited by local and global dead time effects with vastly differing time scales. The global dead time refers to the time interval during which the detector electronics is processing each event. Photons arriving anywhere over the \(256 \times 256\)-pixel area during this time, which is \(1.5 \times 10^{-6}\) sec in our device, are ignored. The local dead time refers to the recovery time for a single microchannel which is saturated on detection of a photon. This dead time, which is \(-10^{-1}\) sec in our device, are ignored. The local dead time effects between detected photons in a single microchannel. As there are three stages in the intensifier, with considerable charge spreading between them, the effect of the local dead-time is not straightforward, though it produced no noticeable effects in the measured space-time correlation function.

The exact effect of the global dead time on the measured correlation is complicated and depends in detail on the statistical properties of the intensity process that is being estimated. Chang et al.\textsuperscript{14} have investigated the dead time affected counting distribution \(p(n)\) for a photomultiplier-based detection system, and their results form the starting point for the present multipixel case. As regards the global dead time, the IPD is, of course, quite different from an array of \(256 \times 256\) photomultipliers: with the IPD, the detection of a photon at any one pixel domain inhibits the detection of photons at all other pixel domains for a global dead time interval, whereas the dead time effects between pixels are independent for an array of separate detectors.

A detailed analysis of the effect of the global dead time on the measured correlation with the IPD will appear in a forthcoming paper.\textsuperscript{15} For the types of dynamic speckle pattern studied in this paper (in which a large number of speckles illuminate the whole detector), it turns out that both \(\langle nm \rangle\) and the product \(\langle n \rangle \langle m \rangle\) are, to first order identically effected, and, therefore, the normalized space–time correlation function is unaffected by the global dead time. However, when the speckle size is comparable to the slit window a noticeable loss of contrast will result at high photon rates along with a modest shape alteration. It must be stressed that this result is only true to first order in \(\tau_d/\Delta t\) and that it only applies to the types of dynamic speckle pattern investigated in this paper.

As a final example of a systematic bias, the effect of spatial and temporal integration on the measured correlation is considered. Our detector area is \(18 \times 18\) mm, which is digitized to a \(256 \times 256\)-pixel array, each pixel domain occupying \(~70 \times 70\) \(\mu\)m\(^2\). In software we can define any combination of these pixel domains to constitute a single detector. Due to the SNR considerations given in Sec. III.C, we used a limited number of detector elements which were, for simplicity, in the form of a rectangular slit window of \(16 \times 110\) pixels in which we defined a linear array of 110 detectors. In other words each detector element was defined as 16 pixels in height by 1 pixel in length. Although the speckle sizes used were always larger than 16 pixels (~1.1 mm), the spatial integration proved to be a noticeable effect. Also discussed here are the (rather minor) effects due to time integration and the effect of spatiotemporal blurring due to the speckle translation across the detector elements. Goodman\textsuperscript{8} has shown that the measured normalized intensity variance \(\sigma^2/(I)^2\) seen by a detector of aperture \(P(\Delta x, \Delta y, \tau)\) in space and time is
\[
\frac{\sigma^2}{\langle I \rangle^2} = \frac{1}{S} \int \int \int \mid P(\Delta x, \Delta y, \tau) \mid \delta(\Delta x, \Delta y, \tau) d\Delta xd\Delta yd\tau,
\]
where
\[
S = \int \int \int \mid P(\Delta x, \Delta y, \tau) \mid \delta(\Delta x, \Delta y, \tau) d\Delta xd\Delta yd\tau,
\]
and \(\delta\) denotes autocorrelation.

The integrating function \(P(\Delta x, \Delta y, \tau)\) is simply the product of three rectangle functions:
\[
P(\Delta x, \Delta y, \tau) = \text{rect}(\Delta x/a) \text{rect}(\Delta y/b) \text{rect}(\tau/\Delta t).
\]
It is straightforward to show that the normalizing constant is just the product of the aperture areas, i.e., \(S = (ab\Delta t)^2\). The correlation function \(C_{\Delta t}(\Delta x, \Delta y, \tau)\) is the product of three Gaussians as shown in Sec. II:
\[
C_{\Delta t}(\Delta x, \Delta y, \tau) = \exp(-\tau^2/\tau_c^2) \exp\left[\frac{-1}{\tau^2} (\Delta x - v_x \tau)^2\right] \times \exp\left[\frac{-1}{\tau^2} (\Delta y - v_y \tau)^2\right],
\]
where \(v_x, v_y\) are the translational speeds along the \(x\) and \(y\) directions in the measurement plane.

A measurement at the center of the diffraction field yields \(v_x = v_y = 0\), which means the speckle simply boils without translating. In this case Eq. (28) separates into three integrals with the solution
\[
\frac{\sigma^2}{\langle I \rangle^2} = f(a/r) f(b/r) f(\Delta t/r_c),
\]
where

1 December 1985 / Vol. 24, No. 23 / APPLIED OPTICS 4215
Fig. 2. Integration function \( f(\alpha) \). The curve indicates how the measured contrast is reduced from the ideal value of one as a function of the space–time integration windows \( \alpha = a/r, b/r, \) or \( \Delta t/\tau_c \).

\[
f(\alpha) = \frac{1}{\alpha^2} [a \sqrt{\alpha} \text{erf}(\alpha) - (1 - \exp(-\alpha^2))],
\]

and \( \text{erf}(\cdot) \) is the error function.

The function \( f(\alpha) \) is shown in Fig. 2. Typical measurements were for \( a/r \approx 0.05, b/r \approx 0.8, \) and \( \Delta t/\tau_c \approx 0.1 \), which yields \( \alpha^2/(\langle I \rangle^2) \approx 0.88. \) Thus the spatial and temporal integration effects in a typical measurement reduce the contrast level to \( \approx 0.88 \) of the maximum, and this can be seen in the data shown in Sec. IV.

A related effect occurs when the speckle is translating across the detector window with velocity \( \mathbf{v} = (v_x, v_y) \). In our measurements we had essentially linear translation along the \( x \) direction; thus \( v_x = v, v_y = 0. \) In this case

\[
C_{\Delta}(x, y, \tau) = \exp(-\tau^2/\tau_c^2)
\times \exp \left[ -\frac{1}{r^2} (\Delta x - v_x \tau)^2 \right] \exp(-\Delta y^2/\tau_c^2).
\]

The measured contrast of Eq. (28) is now a function of the space–time coupling in the \( x \) direction:

\[
\frac{\sigma^2}{\langle I \rangle^2} = f(b/r)g,
\]

where

\[
g = \frac{1}{a^2 \Delta \tau^3} \int_{-\Delta \tau}^{\Delta \tau} \exp(-r^2/\tau_c^2)(\Delta \tau - |\tau|) d\tau
\times \int_{-a}^{a} \exp \left[ -\frac{1}{r^2} (x - v_x \tau)^2 \right] (a - |x|) dx.
\]

This integral reduces to four integrals over error functions and exponentials. To investigate the magnitude of this effect for typical measurement parameters, integral (32) was solved numerically for the fixed parameters \( a/r = 0.05, \Delta t/\tau_c = 0.1, \) and velocities \( 0 \leq v \Delta t/\tau_c \leq 1.75. \) The functional form of \( g \) is quite similar to that of the time-independent function \( f(\alpha) \) as can be seen by comparing Figs. 2 and 3. This is not very surprising; it essentially means that the loss of contrast due to the finite speckle velocity \( v \) has a similar effect to increasing the detector window by an amount \( v \Delta t \) along the direction of motion. For most situations of practical interest the parameters \( (a \pm v \Delta t)/r, b/r, \Delta t/\tau_c \) are all small quantities. In this case we have

\[
g \approx 1 - \left( \frac{\Delta t}{\tau_c} \right)^2 - \frac{1}{6} \left( \frac{\Delta t}{\tau_c} \right)^2 + \frac{\alpha^2}{r^2},
\]

\[
f(\alpha) \approx 1 - \frac{1}{6} \alpha^2: \alpha \ll 1.
\]

Note that the speckle size \( r \) is defined as the \( 1/e \) radius of the coherence area so that

\[
\int_C C_{\Delta}(x, y, 0) dx dy = \pi r^2.
\]

C. Signal-to-Noise Ratio

In Sec. IIIA we defined the quantity [Eq. (21)]

\[
\langle Q \rangle = \frac{1}{N - k} \sum_{i=1}^{N-k} n_i m_{i+1},
\]

where \( n_i \) and \( m_i \) denote the \( i \)th measurement of \( n \) and \( m \), these being the number of detected photons per sample time at two independent pixels in the detector array. \( \langle Q \rangle \) is a good estimate of the correlation function whenever \( N \) is large enough to encompass many realizations of the fluctuating intensity pattern, and the sampling time satisfies \( \Delta t \ll \tau_c. \) The SNR is defined here as

\[
\text{SNR} = \frac{\langle Q \rangle}{\sigma_Q}.
\]

where \( \sigma_Q^2 = \langle Q^2 \rangle - \langle Q \rangle^2. \) This quantity has been evaluated elsewhere\(^1\) with the result

\[
\text{SNR} = [M(1 + C_{\Delta})]^{1/2}(n),
\]

The functional dependence of \( g \) on these parameters is strictly through the ratios \( a/r, v \Delta t/\tau_c, \) and \( \Delta t/\tau_c. \) The curve is plotted as a function of the variable \( v \Delta t/\tau_c, \) which is the ratio of the distance traversed by a speckle during the integration time \( \Delta t \) to the correlation length \( \tau \) of the speckle.
provided \( \langle n \rangle \ll 1 \). Here, \( M \) is the number of statistically independent realizations of the fluctuating intensity pattern, \( \langle n \rangle \) is the average number of detected photons per sample time per pixel, and \( C_{\Delta I} \) is the normalized correlation of intensity fluctuations \((0 \leq C_{\Delta I} \leq 1)\). If the detector array consists of \( N_x \times N_y \) detectors, the correlator measures \( N_x N_y \) independent estimates of \( \langle Q \rangle \) at the spatial lag \((0,0)\). In general, the correlator measures \( N(A_x,A_y) = N_x N_y \) independent estimates of \( \langle Q \rangle \) at the arbitrary spatial lag \((A_x,A_y)\), where

\[
G(A_x,A_y) = (1 - |A_x|/N_x)(1 - |A_y|/N_y) \quad 0 \leq |A_x| \leq N_x - 1, \quad 0 \leq |A_y| \leq N_y - 1, \quad (37)
\]

is the triangle function form Sec. III.A. Therefore, the measured correlation function has SNR distribution, which depends on the vector \((A_y,A_y)\):

\[
\text{SNR}_{\Delta x,\Delta y} = [MN(\Delta x,\Delta y)(1 + C_{\Delta I})]^{1/2} \langle n \rangle. \quad (38)
\]

For our measurements, \( N_x \sim 110, N_y = 1 \), \( \langle n \rangle \sim 0.3 \) photons/detector/sample time, and \( M \sim 100 \). Using these values we find that

\[
\text{SNR}_{\Delta x} \sim 3[N(\Delta x,0)]^{1/2}
\]

for our experiments. Typical values of the SNR at various spatial lags are given:

| \(|\Delta x|\) | SNR |
|----------------|-----|
| 0              | \~31 |
| 90             | \~15 |
| 110            | \~3  |

Thus the spatial correlation at fixed time lag \( \tau \) has a slowly decreasing value of the SNR for \( 0 \leq |\Delta x| \leq (0.8)N_x \). The presence of the triangular bias imposed on the SNR is in practice not very restrictive as 80% of the measured correlation function has a SNR greater than half of the maximum value at the origin \( \Delta x = 0 \). The anomalous situation at \( \tau = 0 \) is again not restrictive in practice. Since the differencing algorithm does not difference any vector from itself we find at \( \tau = 0 \) that there are \~1/2 of the number of differences generated over the spatial domain \(-N_x \leq \Delta x \leq N_x\) than at \( \tau \geq 1 \). Hence the SNR at \( \tau = 0 \) is \~50% more noisy than at \( \tau \geq 1 \), and this is a modest decrease. In principle there is also a triangular bias in time, but, given a maximum measured time lag which satisfies \( \Delta t_{\text{max}}/\Delta t \leq 10^{-2}N \), this is a negligible effect.

IV. Experimental results

The dynamic speckle experiment is shown in Fig. 1. A 0.5-mW He–Ne laser of beamwidth \( W = 0.45 \) mm illuminates a rotating ground glass diffuser producing dynamic Gaussian speckle in the diffraction plane. The beam is centered on the point \((0,R,0)\) in the scattering plane, and the detector window is centered on \((0,p,z)\) in the diffraction plane. The IPD is housed in a lightproof box \~20 cm in length. The detector window consists of a horizontal slit of dimensions \~1 \times 12 mm, which is mounted on the front end of the box, and the slit is imaged onto the photocathode at 1:1 magnification using an \( f/4 \) aplanatic doublet. The detection geometry ensures that the speckle translation is approximately linear across the slit window. The diffuser rotates with constant angular speed \( \omega \), and the trans-
The transversal speed of the speckle across the slit is given by \( v_{sp} = \omega P x \). The decorrelation time is \( \tau_c = W/\omega R \) as discussed in Sec. II.

For this geometry the spatiotemporal correlation function has the form

\[
C_{a}(\Delta x, \tau) = \exp\left(-\frac{\omega^2 R^2 \Delta x^2}{W^2}\right) \exp\left[-\frac{1}{\tau^2} (\Delta x - \rho \tau)^2\right],
\]

where \( \tau = \lambda z/\pi W \) is the speckle size, and we consider only 1-D motion (hence \( \Delta y = 0 \)). The range of the various experimental parameters is

\[
\lambda = 633 \text{ nm},
\]

\[
\Delta t = 0.0128 \text{ sec}, 0.0256 \text{ sec},
\]

\[
1 \text{ m} \leq z \leq 2.5 \text{ m},
\]

\[
1 \text{ cm} \leq R \leq 2 \text{ cm},
\]

\[
0 \leq \rho \leq 12 \text{ cm},
\]

\[
0.025 \text{ sec}^{-1} \leq \omega/2\pi \leq 0.125 \text{ sec}^{-1}.
\]

Figure 4 shows the time evolution of the correlation function from a moving speckle pattern with parameters \( \rho = 9 \text{ cm}, R = 1.5 \text{ cm}, z = 2.5 \text{ m}, \omega/2\pi = 0.05 \text{ sec}^{-1} \). The deterministic translation is evident from the time dependence of the shifted peaks. The measured translational speed along \( x \) in this case was \( v_x \approx 3.0 \text{ cm/sec} \sim \omega P \) as predicted by theory. The width of each peak is a measure of the speckle size \( r = \lambda z/\pi W \). The measured spatial correlation functions are reasonably Gaussian in shape and maintain the same width for all time lags \( \tau \).

![Figure 5](image)

**Figure 5.** Spatial correlation function measured at the fixed time lag \( \tau = \Delta t \) for two different speckle sizes: (a) \( r_{sp} \approx 0.8 \text{ mm} \); (b) \( r_{sp} \approx 1.5 \text{ mm} \). The parameters for both measurements are the same as in Fig. 4 except that \( z = 1.4 \text{ m} \) in (a). The correlation functions are reasonably Gaussian in shape, although the measured widths are \( \sim 20\% \) larger than the theoretical values \( R_{sp} = (\lambda z)/(\pi w) \). Note that the curves are not centered on the zero spatial lag since \( \tau = \Delta t \).

Figure 5 shows measurements of the spatial correlation function at time lag \( \tau = 1 \) for two different speckle sizes. The measured values of \( r \) are slightly larger than predicted by theory, and one can observe how the contrast level decreases for the narrower speckle size due to spatial integration effects. Figure 6 shows a plot of the peak values of the correlation function in \( \Delta x \) as a function of the time lag \( \tau \) for seven different diffuser velocities. The slope of each line gives the direction and speed of the speckle moving across the slit window for various parameters of \( \rho \) and \( \omega \). The translation is clearly linear, and the experimental values for velocity compare well with the predictions of Sec. II. Figure 7 shows experimental measurements of the temporal decorrelation for two different combinations of \( \omega \) and \( R \). This plot defines \( C(t) \) normalized to the measured variance at \( \tau = 0 \). The Gaussian shape and time scales \( \tau_c = W/\omega R \) are in excellent agreement with theory. The true variance is lowered by a constant factor of \( \sim 0.84 \) due largely to spatial integration effects discussed in the previous section.

As a final example we show measurements of the spatial correlation structure of a double speckle pattern using the setup shown in Fig. 8. In this configuration we use two 0.5-mW He–Ne lasers which intersect at the diffuser surface forming a small angle \( \theta \). The diffraction plane consists of two virtually identical speckle patterns which are shifted by the angle \( \theta \). This arrangement simulates the characteristic double speckle pattern generated by a binary star system viewed through a turbulent atmosphere\(^{18} \) with a ground-based telescope. In this case the intensity distribution due to the double speckle pattern has the form using a 1-D notation

\[
I(x) = I_1(x) + I_2(x + \delta),
\]

where \( \delta = \omega P x/\pi R \) is the translation between the two speckle patterns. Figure 6 shows experimental measurements of the temporal decorrelation for a double moving across the slit windows. The relevant parameters are, from left to right,

\[
x: \rho = 70 \text{ mm}, \omega = -2\pi/20 \text{ sec}^{-1} \quad R = 13 \text{ mm},
\]

\[
a: \rho = 87 \text{ mm}, \omega = -2\pi/60 \text{ sec}^{-1} \quad R = 13 \text{ mm},
\]

\[
c: \rho = 65 \text{ mm}, \omega = -2\pi/60 \text{ sec}^{-1} \quad R = 35 \text{ mm},
\]

\[
w: \rho = 5 \text{ mm}, \omega = -2\pi/60 \text{ sec}^{-1} \quad R = 25 \text{ mm},
\]

\[
s: \rho = 70 \text{ mm}, \omega = +2\pi/60 \text{ sec}^{-1} \quad R = 30 \text{ mm},
\]

\[
r: \rho = 87 \text{ mm}, \omega = +2\pi/60 \text{ sec}^{-1} \quad R = 13 \text{ mm},
\]

\[
o: \rho = 70 \text{ mm}, \omega = +2\pi/18 \text{ sec}^{-1} \quad R = 10 \text{ mm}.
\]
Fig. 7. Measurement of the temporal decorrelation for dynamic speckle with parameters: (a) $R = 13\, \text{mm}$, $\omega = 2\pi/14\, \text{sec}^{-1}$, $\rho = 75\, \text{mm}$; and (b) $R = 15\, \text{mm}$, $\omega = 2\pi/30\, \text{sec}^{-1}$, $\rho = 65\, \text{mm}$. The + signs refer to the peak measured variance of the spatial correlation function normalized to the measured variance at $r = 0$ and are plotted for time lags $0 \leq r \leq 9\Delta t$. The solid line is the theoretical curve $C_{\Delta t}(r) = \exp(-r^2/\tau^2)$, where $\tau = W/R\omega$.

where $\delta$ is the linear separation at the detection plane, and $I_1$ and $I_2$ are the intensities of the two distributions. Thus

$$C_{\Delta t}(\Delta x) = \frac{\langle [I(x + \Delta x) - I(x)]^2 \rangle - \langle I(x) \rangle^2}{\langle I(x) \rangle^2}$$

(41)

$$= \frac{(1 + \mu^2) \exp(-\Delta x^2/\tau^2) + \mu[\exp(-(\Delta x + \delta)^2/\tau^2) + \exp(-(\Delta x - \delta)^2/\tau^2)]}{(1 + \mu^2)}$$

where $\mu = \langle I_2 \rangle / \langle I_1 \rangle$. For our case, $\mu = 1$, and, therefore,

$$C_{\Delta t}(\Delta x) = \frac{1}{2} \exp(-\Delta x^2/\tau^2) + \frac{1}{2}[\exp(-(\Delta x + \delta)^2/\tau^2) + \exp(-(\Delta x - \delta)^2/\tau^2)]$$

(42)

The variance at the origin ($\Delta x = 0$) is twice that of the side peaks (assuming $\delta > \tau$) and lower by a factor of 2 from the single speckle pattern situation.

The measurements shown in Fig. 9 are for parameters $r = 0.6\, \text{mm}$, $\delta = 2.5$, and $3.5\, \text{mm}$. The contrast

Fig. 8. Experimental arrangement for the simulated binary star system as viewed through a turbulent atmosphere. The two beams intersect at the scattering plane where a moving diffuser is located and the detection plane consists of two identical speckle patterns which are shifted by an angle $\theta$ as shown.

Fig. 9. Measurements of the spatial correlation function for the geometry of Fig. 8. The angles are (a) $\theta = 2.5 \times 10^{-3}\, \text{rad}$, (b) $\theta = 3.5 \times 10^{-3}\, \text{rad}$, and $\tau = 1$ for both measurements.
ratio of the central peak to either of the side peaks is in excellent agreement with Eq. (42). Due to constraints imposed by the slit window size and the small angles required, the detector was placed at \( z = 1 \) m from the diffuser, and the resulting speckle size is the order of the slit width. This in turn gives rise to a large spatial integration effect causing a 50% overall reduction in the measured contrast. Despite all this, however, the correlation function was easily and consistently measured with very little distortion.

V. Conclusions

It has been demonstrated that the dynamic correlation structure of a fluctuating intensity pattern may be accurately measured simultaneously in the space and time domains using a new multipoint photon detector. There are a number of systematic biases present in the data, but these can be compensated for. The present method of data processing uses a desktop computer, which is \( \sim 200 \) times too slow to keep up with a typical data rate of \( 10^4 \) detected photons/sec. However, it is possible, using currently available integrated circuits, to construct a hard-wired correlator based on the same algorithm for calculating the correlation function.

References

1. This detector was purchased from Instrument Technology, Ltd., 29 Castleham Rd., St. Leonards-on-Sea, East Sussex, TN38 9NS, U.K.
11. Ref. 9, p. 235.
13. Ref. 12, p. 121.
16. Ref. 10, p. 46.

The authors wish to thank Tom Gonsiorowski and Kevin O'Donnell for their generous technical assistance and for many stimulating discussions. They also wish to thank a referee for correcting the original version of this paper. This research was supported by a grant from the U.K. Science and Engineering Research Council (GR/C 78940).