Object reconstruction from photon-limited centroided data of randomly translating images

L. C. de Freitas* and J. C. Dainty

Blockett Laboratory, Imperial College, London SW7 2BZ, UK

Received November 23, 1987; accepted January 25, 1988

Centroiding is investigated as a simple and computationally fast technique of object reconstruction, at low light level, from randomly translating images. A relationship between the spectrum of the average N-photon centroided frame and the object spectrum is presented as well as an algorithm for retrieving the phase in the case of one-dimensional objects. Computer-simulated data are used to test the relationship and the reconstruction algorithm.

A number of techniques have been developed to retrieve diffraction-limited astronomical images by recording short-exposure frames of speckle patterns. The shift-and-add (SAA) method1-5 is one of the proposed techniques. It relies on the proposition that each frame consists of many distorted replicas of the true image and that an improved estimate of the image can be found by superimposing these distorted replicas. At very low light levels each of these distorted images may have only a few photons, and no bright speckle can be chosen in order to implement the SAA technique.

Centroiding is investigated as a new way of circumventing this limitation, and for this purpose a relationship between the N-photon averaged centroided images and the stationary normalized image $I(r)$ as well as a phase-reconstruction algorithm is presented for the special case of randomly translating images of one-dimensional objects.

To freeze a randomly moving image, photons, which are all assumed to emanate from the same randomly translating image, are detected during a series of short time intervals (frames). To retrieve the stationary image one should, for each frame $k$, shift the photon vectors by the amount by which the true centroid of the image is displaced in respect to the center of the frame and average (add) over all the frames. The true shift vector $\mathbf{c}_k$ is unknown; as an estimator the centroid vector $\mathbf{R}_k = (1/N) \sum_{i=1}^{N} x_i$ of the detected photons is evaluated, and the photon vectors are shifted by this estimator before averaging many such frames.

The relationship between the centroided and the noncentroided spectrum of the N-photon data $d_k(x)$ detected on the $k$th frame is

$$\tilde{D}_k^c(u, x_1, \ldots, x_N) = \exp(+i2\pi u \cdot \mathbf{R}_k) \tilde{D}_k(x, x_1, \ldots, x_N),$$

and the average of $\tilde{D}_k^c(u, x_1, \ldots, x_N)$ over all possible sets of detected N-photon coordinates is performed, using the normalized object intensity as the probability-density distribution. Thus Eq. (1) leads to a quantity $\tilde{Q}_N(u) = \tilde{D}_k^c(u)/N$ related to the normalized object intensity spectrum:

$$\tilde{Q}_N(u) = \frac{I([u(1-1/N)]/(-u/N)]N^{-1}}{N}.$$  \hspace{1cm} (2)

Equation (2) is a translation-invariant relationship between the normalized spectrum of the stationary image $I(u)$ and the spectrum $\tilde{Q}_N(u)$ of the centroided average of those frames containing exactly $N$ photons per frame. For instance, for $N = 2$, Eq. (2) reduces to $\tilde{Q}_2(u) = |I(u/2)|^2$, which is the power spectrum of $I(u)$, where the variable $u$ is scaled by a factor of 2.

For $N = 3$, $\tilde{Q}_3(u) = I\left(\frac{2}{3}u\right)I\left(-\frac{u}{3}\right)I\left(-\frac{u}{3}\right), \hspace{1cm} \hspace{1cm} (3)$

which is an expression related to the bispectrum7 when it is evaluated at frequencies $u \rightarrow (2/3)u, v \rightarrow -(1/3)u$ or when $u \rightarrow -(1/3)u, v \rightarrow (2/3)u$, both solutions being representations of lines in the bispectrum $u, v$ plane. It can also be seen in the limiting case of $N \rightarrow \infty$ that $\tilde{Q}_N(u) \rightarrow I(u)$, as one would expect.

In Eq. (2) if one changes $u/N \rightarrow v$ and considers the case of real one-dimensional intensity $[I(-u) = I^*(u)]$, one has

$$\tilde{Q}_N(Nu) = I((N-1)u)/[I^*(u)]N^{-1}. \hspace{1cm} \hspace{1cm} (4)$$

From Eq. (4) follows a recurrence relation linking phases of the object, $\phi$, at the discrete frequencies $k$ and $[N-1]k$ with the phase $\theta_N(Nk)$ of the average quantity $\tilde{Q}_N(Nk)$:

$$\theta_N(Nk) = \phi([N-1]k) - (N-1)\phi(k).$$ \hspace{1cm} (5)

For $N = 2$ photons/frame, $\theta_2(2k) = \phi(k) - \phi(k) = 0$, and therefore no information about the phase can be retrieved. But for $N = 3$ photons/frame, Eq. (5) becomes

$$\theta_3(3k) = \phi(2k) - 2\phi(k), \hspace{1cm} (6)$$

and hence from the phase at frequency $k$ one can reconstruct the phase at $2k$ up to $3k \leq (L/2) - 1$, where $L$ is the actual number of bins used to sample $Q_3(r)$.

The method can be better understood through an example in which a one-dimensional image is sampled at 32 points and therefore $\tilde{Q}_N(Nk)$ is determined only

0146-9592/88/040264-03$2.00/0 © 1988, Optical Society of America
Fig. 1. Pictorial representation of the phase-reconstruction algorithm for $N = 3$ and $N = 4$ photons/frame (ph/fr).

(a) Stationary Object

(b) Reconstruction $N=3..13$

Fig. 2. (a) Stationary object intensity, (b) reconstruction with $N$ ranging from 3 to 13 photons/frame.

at 32 frequency bins. Because $\tilde{I}(-u) = \tilde{I}^*(u)$, one has to find the phases only in half of the total number of bins and then reverse their sign. In this example one has, therefore, to consider only 16 frequency bins (Fig. 1).

For $k = 0$ the phase $\phi(0) = 0$. As $Q_N(Nk)$ is shift invariant, one can always determine $\tilde{I}(k)$ apart from a constant phase-shifting factor. By appropriately choosing this factor one can always set the phase at $\phi(1) = 0$.

In general, for a frequency bin $k$, where $k$ is a prime number, one can reconstruct its phase only by using an average $Q_N(Nk)$ such that $N = k + 1$ photons per frame. In this example, to reconstruct the phases up to bin 15 one needs averages of frames with $N$ as many as 16 photons per frame. The modulus of $\tilde{I}(k)$ is found from the Fourier transform of the autocorrelation function, the power spectrum $|\tilde{I}(u)|^2$.

Experiments were carried out using simulated one-dimensional photon data emitted from the object shown in Fig. 2(a), and in the simulation experiment described in this Letter we set the Poisson-distribution mean value $N = 3$ photons/frame. Frames with 0 or 1 photons/frame are disregarded, because they do not carry any information concerning the intensity distribution of the image, and a total of 80,084 frames of a randomly translating image (of a binary star) containing $N \geq 3$ photons/frame were generated. These frames can be grouped in sets of frames containing a number of photons ranging from $N = 3$ up to $N = 13$ photons/frame in this case.

Figure 3 depicts $Q_N(r)$ for $N = 2$ and $N = 3$ photons/frame. For $N = 2$ it can be seen that $Q_2(r)$ exhibits the same shape as the autocorrelation of the binary but distributed in a smaller region of space.

Figure 2(b) shows the reconstructed image obtained by using this method with $\mathcal{L} = 128$ bins and using frames with $N$ only 13 up to photons/frame, i.e., not all the frequencies were reconstructed. Despite the fact that no other image-processing technique has been used, e.g., enforcing positivity, Fig. 2(b) nevertheless

Fig. 3. (a) Centroided frames with $N = 2$ photons/frame, (b) centroided frames with $N = 3$ photons/frame.
gives a good estimate of the image and in this case also a good estimate of the relative brightness of the stars. As described in this Letter, the technique may be useful in applications such as tracking fast-moving objects viewed through telescopes. We are currently investigating the extension of this method to imaging through turbulence.

The research of L. C. de Freitas is funded by The British Council and the Conselho Nacional de Desenvolvimento Científico e Tecnológico—Brazil. The authors wish to thank G. Ayers for providing the simulation program and also to acknowledge the contributions of M. J. Northcott and G. Ayers in the early stages of this work. The research was supported by the UK Science and Engineering Research Council (SERC) under grant GR/D 92332 and by the U.S. Army under contract DAJA 45-85-C-0028. J. C. Dainty is a SERC Senior Research Fellow.

* On leave from the Departamento de Física, Universidade Federal Mato Grosso do Sul, P.O. Box 649, 79.100 Campo Grande, MS, Brazil.

References