Multiple scattering from random rough surfaces using the Kirchhoff approximation

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Abstract. Numerical results for scattering of electromagnetic waves from a rough surface are presented and compared with experimental results. The method uses the Kirchhoff or physical optics approximation to separate the single and double scatter terms in the total scatter pattern. It is shown that enhanced backscatter occurs in the double scatter term as predicted by a simple ray picture of the scattering process.

1. Introduction

The observation of enhanced backscattering by randomly rough surfaces of large root-mean-square (r.m.s.) slope [1–4] has stimulated a re-examination of theoretical approaches to scattering by optically rough surfaces [5–18]. It is now possible, at least for a one-dimensional surface, to solve Helmholtz' equation numerically, realisation-by-realisation, to find the mean scattered intensity and its statistical properties (e.g. variance). This is a powerful method for predicting the angular distribution of the scattered intensity and has been exploited by Maradudin [8–10], Maystre [11], Nieto-Vesperinas [12–14], Thorsos [17] and others. Using this method one can predict the scattered intensity in situations where controlled experiments are difficult or tedious.

The agreement between these exact calculations and experiment, e.g. as reported in [4], is good, but not perfect. The problem with the exact method is that it gives little physical insight into the scattering process. For example, the exact method does not answer the simple question: what is the role of multiple scattering for these high-sloped surfaces? In contrast, the literature on light scattering by dense volume media [19–21], emphasizes the underlying physical processes such as multiple scattering.

The purpose of this paper is to use an approximate theory based on the Kirchhoff approximation to investigate the role of multiple scattering from randomly rough surfaces of large r.m.s. slope. A simple picture of the scattering process [1, 2] leads to the conclusion that the backscatter peak is caused by multiple scatter paths. From figure 1 the scattering paths ABC and CBA give scattered waves with a difference in phase

\[(k_1 + k_2) \cdot AC.\]

This term is zero if

\[k_2 = k_1,\]

which is exactly the case for backscatter. Thus the backscatter terms add coherently to give a factor of two larger intensity in this direction than in other directions which...
Figure 1. The geometry for the simple ray model of the scattering process showing a possible path ABC and its time reversed partner CBA.

have incoherently added terms. The width of the backscatter peak can also be estimated from this simple model. The coherent term will give no contribution when the scattered waves are $\pi$ out of phase. The half width of the peak is then found to be

$$\theta_{1/2} \approx \frac{\lambda}{\langle|AC|\rangle}.$$ 

To calculate the field scattered by a surface, first the boundary conditions on the field are used to find the field on the surface and then the scattered (usually far field) distribution of this field is found. There are two categories of solution: those based on exact boundary conditions \cite{5,6,8-16,22-28} and those based on approximate boundary conditions \cite{7,17,18,29-33}.

When exact boundary conditions are used, the solution has to be found either numerically as in \cite{5,6,8-16}, or approximately using an appropriate expansion for the scattered field. The problem then is that it is generally very difficult to have a good physical picture of what the equations represent since the expansions of the scattered field tend to be rather complicated. This means that the methods do not give much insight into the physical mechanisms behind the resulting scattered intensity distributions, in particular whether the backscatter intensity is due to single or multiple scattering terms.

The second group of methods, in particular the Kirchhoff or physical optics methods, can however give a clear picture of the physical processes involved in the scattering. In the physical optics method the approximate boundary conditions at each reflection point are taken to give, in the two-dimensional case

$$E_i(x, z) = (1 + R)E_i(x, z), \quad (1)$$

$$\frac{\partial E_i(x, z)}{\partial n} = i(1 - R)k \cdot \mathbf{n}E_i(x, z), \quad (2)$$

$$E_i(x, z) = (1 + R)E_i(x, z), \quad (1)$$

$$\frac{\partial E_i(x, z)}{\partial n} = i(1 - R)k \cdot \mathbf{n}E_i(x, z), \quad (2)$$
where $E_i(x, z)$ is the total field at the point $(x, z)$ which is on the surface, $R$ is the planar reflection coefficient at that point which depends on the local incidence angle, $E_i(x, z)$ is the field incident at that point, $k_i$ is the incident wave-vector and $n$ is the outward normal to the surface at the point $(x, z)$. This simply states that the total field on the surface is the sum of the incident and reflected field at each point.

The scattered field is usually taken to be given by the two-dimensional Helmholtz integral equation, including the shadowing functions

$$E_s(x, z) = \frac{1}{4i} \int_S S(x', z') S'(x', z') \left( E_i(x', z') \frac{\partial H_0^{(1)}(kr)}{\partial n} - H_0^{(1)}(kr) \frac{\partial E_i(x', z')}{\partial n} \right) ds', \quad (3)$$

where $r = [(x - x')^2 + (z - z')^2]^{1/2}$, $H_0^{(1)}(kr)$ is the zeroth order Hankel function of the first kind, $ds'$ is an element of the surface and $S(x', z')$ and $S'(x', z')$ represent the incidence and scatter shadow functions respectively,

$$S(x', z') = \begin{cases} 1, & \text{if } (x', z') \text{ is illuminated,} \\ 0, & \text{if } (x', z') \text{ is not illuminated,} \end{cases}$$

$$S'(x', z') = \begin{cases} 1, & \text{if } (x', z') \text{ is visible,} \\ 0, & \text{if } (x', z') \text{ is not visible.} \end{cases}$$

It is important to note that these are geometrical shadowing functions and the effects of diffraction are not taken into account. The use of these straight line functions is reasonable if the distance between the blocking edge and the shadowed point is only a few wavelengths, i.e. if the shadowed point is in the Fresnel region of the blocking point, or if the distance between the source point and the blocking point is of the same order. This situation was satisfied in the cases studied with the distances being of the order of the correlation length of the surface which was typically $3\lambda$.

The effects of the shadowing functions were discussed by Brown [33] who deduced that the incidence shadowing function was valid for the case of single scatter whereas the scatter shadow function showed unphysical behaviour. Brown suggested that the scatter shadow function causes the surface field to depend on the position of the observer when it should depend only on the incidence angle and the form of the surface profile. However, $S'(x', z')$ does not change the surface field, it changes the observed surface field, or rather it changes the parts of the surface (and hence the surface field distribution) which are visible. This is not unphysical. The main problem with $S'(x', z')$ can be seen from figure 1. The light scattered from point A in the direction $\hat{1}_{AB}$ is blocked at point B and is re-directed. This means $S'(x', z') = 0$ and the contribution from this light is neglected, i.e. the re-directed light is ignored. Therefore $S(x', z')$ in equation (3) is valid for the case of single scatter.

Hence investigations of the validity of equation (3) are therefore investigations of the validity of two separate approximations.

(i) that the total surface field and its normal derivative at each point are given by equations (1) and (2) at each reflection and

(ii) that the total scattered field can be approximated by the single scatter term of equation (3).

In this work only the first approximation is used; the second is not required as the second order, double scatter, term is explicitly calculated. This involves using (i) at a scatter point, propagating to another point on the surface taking shadowing into account, using (i) again at the second scatter point and then propagating to the far
zone to give the scattered field. This term is calculated for every pair of points to give the total double scatter contribution.

Obviously this second-order term suffers from the same problem as equation (3), in that there will be a scatter shadow function which will predict that light is blocked and re-directed but then that light will be ignored. However the energy carried by this light is much smaller than that carried by the double scatter term and it is reasonable to expect that any further contribution from this light will be very small. The third-order or triple-scatter has also been calculated for a few cases and is found to be negligibly small.

The surfaces considered in this work have a Gaussian probability density of height and a Gaussian correlation function. Such surfaces are manufactured in our laboratory [3, 4] using the technique of exposing a plate with a layer of photo-resist to many independent speckle patterns and then coating with gold. The experimental results used are normalised to unit area beneath the curves i.e. the material dependence of the total scattering cross-section has been removed.

There remains the question of validity of the solution obtained. Equations (1) and (2) contain the planar reflection coefficients so the solution should be valid for surfaces which are locally quite flat. This requirement is usually interpreted that the radius of curvature of the surface should be very much less than the wavelength of the incident illumination. The radius of curvature is approximately the inverse of the second derivative of the surface height distribution. As the height distribution is gaussian, so is the second derivative, with a standard deviation

$$\sigma_{h''} = 2\sqrt{3} \frac{\sigma_h}{\tau^2},$$

where $\sigma_h$ is the standard deviation of the surface height distribution and $\tau$ is the $1/e$ correlation length. One of the surfaces for which data was available [4] was surface 440 with $\sigma_h \approx 1.2$ µm and $\tau \approx 2.9$ µm, which gives $\sigma_{h''} \approx 0.5$ µm$^{-1}$ and thus a radius of curvature of 2 µm. Considering illumination with $\lambda = 0.633$ µm (He–Ne) should give a case where most of the radii of curvature are greater than the wavelength and the approximation (i) is reasonably valid.

2. Theory

The single scatter term is given by equation (3), where the observation point has been taken to be in the far zone. Using the far fields of the Hankel functions

$$\lim_{r \to \infty} H_0^{(1)}(kr) = \left(\frac{2}{\pi kr}\right)^{1/2} \exp \left[i \left(kR_0 - \frac{\pi}{4} - k \cdot R\right)\right],$$

$$\lim_{r \to \infty} \frac{\partial H_0^{(1)}(kr)}{\partial n} = -\left(\frac{2}{\pi kr}\right)^{1/2} i k \cdot n \exp \left[i \left(kR_0 - \frac{\pi}{4} - k \cdot R\right)\right],$$

and equations (1) and (2) gives

$$E_s(x, z) = \left(\frac{2}{\pi kr}\right)^{1/2} \frac{\exp i \varphi}{4} \int \int S(x', z')S'(x', z')[(1 + R)E_i(x', z')k \cdot n \exp (-ik \cdot R)

- (1 - R)E_i(x', z')k \cdot n \exp (-ik_1 \cdot R)] ds',$$
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Figure 2. The geometry used for the Kirchhoff approximation. Note that the incident and scattered angles are measured in opposite senses.

where $\varphi$ is $kR_0 - \pi/4$, $R_0$ is the vector $(x, z)$ and $R$ is the vector $(x', z')$. Also (see figure 2).

\[ k_i = k \sin \theta_i x - k \cos \theta_i z, \]
\[ k = k \sin \theta x + k \cos \theta z, \]
\[ n = -\sin \beta x + \cos \beta z, \]
\[ ds = \frac{dx}{\cos \beta}, \quad \tan \beta = \frac{dh(x)}{dx} = m. \]

Then

\[ E_s(0) = -\left( \frac{2}{\pi kr} \right)^{1/2} \exp i\varphi \frac{4}{4} \int_{R} S(x', z')S'(x', z')[(1 + R)k(m \sin \theta - \cos \theta)
+ (1 - R)k(m \sin \theta_i + \cos \theta_i)]E_i(x', z') \exp (-ik \cdot R) dx'. \quad (7) \]

The incident field is given by a plane wave

\[ E_i(x', z') = A_i \exp (ik \cdot R). \quad (8) \]

Therefore, following Beckmann, equation (7) reduces to

\[ E_s(0) = -\left( \frac{2}{\pi kr} \right)^{1/2} \frac{\exp i\varphi}{4} \left[ \frac{1 + \cos(\theta + \theta)}{\cos \theta + \cos \theta_i} \right] \]
\[ \times \int_{R} S(x', z')S'(x', z')RA_i \exp i(k - k) \cdot R dx' + e(\theta), \quad (9) \]
where \( e(\theta) \) is an edge effect term which is much smaller than the first term in equation (9) and so shall be neglected.

Now the surface is discretised into \( N \) segments to simplify the integration. The phase term inside the integral is taken to be constant over each segment. This is simply a condition on the number of segments along the length of the surface, the length being small enough to neglect the phase term. Taking this and the shadow terms to have the value at the centre of each segment gives

\[
E_5(\theta) = -\left( \frac{2}{\pi kr} \right)^{1/2} \exp \left( \frac{i \phi}{4} \right)^2 \left[ 1 + \cos (\theta + \theta_i) \right] \cos \theta + \cos \theta_i
\]

\[
\times \sum_{j=1}^{N} RS(x'_j, z'_j)S'(x'_j, z'_j) \exp[i(k_i - k_j) \cdot R_j] \int_{(j-1)\Delta x}^{j\Delta x} A_i \, dx',
\]

with the surface split into \( N \) segments of equal x-distance \( \Delta x \). Assuming a small enough value of surface segment length \( \Delta x \) gives

\[
\int_{(j-1)\Delta x}^{j\Delta x} A_i \, dx' = A_i \Delta x.
\]

Considering an incident plane wave of unit amplitude, \( A_i = 1 \), finally gives

\[
E_5(\theta) = -\left( \frac{2}{\pi kr} \right)^{1/2} \exp \left( \frac{i \phi}{4} \right)^2 \left[ 1 + \cos (\theta + \theta_i) \right] \cos \theta + \cos \theta_i
\]

\[
\times \sum_{j=1}^{N} RS(x'_j, z'_j)S'(x'_j, z'_j) \exp[i(k_i - k_j) \cdot R_j] \Delta x.
\]

This is the usual single scatter term taken to be the solution for the physical optics approximation.

Now consider the double scatter contribution. To find this term the field obtained by scattering the incident field from one point on the surface to another point on the surface must be found, and then this field at any surface point due to light scattered from all the other surface points is

\[
E_s(x_2, z_2) = -\frac{1}{4i} \int S(x_1, z_1)S_{12} \left\{ (1 + R_1) \left[ m_1 \frac{k(x_2 - x_1)}{r_{12}} - \frac{k(z_2 - z_1)}{r_{12}} \right] H_1^{(1)}(kr_{12}) \right. \\
- (1 - R_1)ik(m_1 \sin \theta_1 + \cos \theta_1)H_0^{(1)}(kr_{12}) \left\} E_i(x_1, z_1) \, dx_1,
\]

where the subscript 2 represents the second point and subscript 1 the first point, \( r_{12} = [(x_2 - x_1)^2 + (z_2 - z_1)^2]^{1/2} \), the normal derivative of the Hankel function is given by

\[
\frac{\partial H_0^{(1)}(kr_{12})}{\partial n} = \left[ n_x \frac{k(x_2 - x_1)}{r_{12}} + n_z \frac{k(z_2 - z_1)}{r_{12}} \right] H_1^{(1)}(kr_{12}),
\]

and the shadow function \( S_{12} \) is defined as

\[
S_{12} = \begin{cases} 
1, & \text{if } (x_2, z_2) \text{ is visible from } (x_1, z_1), \\
0, & \text{if } (x_2, z_2) \text{ is not visible from } (x_1, z_1).
\end{cases}
\]
The field scattered from all these second points will then be the double scatter term

\[ E^{(2)}_s(\theta) = -\left( \frac{2}{\pi kr} \right)^{1/2} \frac{\exp(i\varphi)}{4} \int_r S'(x_2, z_2) \left[ (1 + R_2)k(m_2 \sin \theta - \cos \theta)E_s(x_2, z_2) + i(1 - R_2) \frac{\partial E_s(x_2, z_2)}{\partial n_2} \right] \exp(i \mathbf{k} \cdot \mathbf{R}_2) \, dx_2. \]  

(14)

Equations (12) and (14) together form the double-scatter term. The general form of this term can be seen to be rather complicated. However, if p-polarised (TM) light incident on a perfectly conducting rough surface is considered, \( R = 1 \) for both points 1 and 2. Then writing

\[ S'_{12} = S(x_1, z_1)S_{12}S'(x_2, z_2), \]


\[ E^{(2)}_s(\theta) = -\left( \frac{2}{\pi kr} \right)^{1/2} \frac{\exp(i\varphi)}{4i} \int_r \int_r S'_{12} \left[ m_1 \frac{k(x_2 - x_1)}{r_{12}} - \frac{k(z_2 - z_1)}{r_{12}} \right] \]

\[ \times H_1^{(1)}(kr_{12})k(m_2 \sin \theta - \cos \theta) \exp \left[ i(\mathbf{k}_1 - \mathbf{k}) \cdot \mathbf{R}_2 \right] dx_2 \, dx_1. \]  

(15)

For the discretisation in this case if it is assumed that \( dx \) is small enough compared to \( \lambda \) that it is possible to simply replace each integral with the sum of the values at the centre point of each segment multiplied by \( \Delta x \)

\[ E^{(2)}_s(\theta) = -\left( \frac{2}{\pi kr} \right)^{1/2} \frac{\exp(i\varphi)}{4i} \sum_{j=1}^{N} \sum_{i=1}^{N} S'_{ji} \left[ m_j \frac{k(x_i - x_j)}{r_{ij}} - \frac{k(z_i - z_j)}{r_{ij}} \right] \]

\[ \times H_1^{(1)}(kr_{ij})k(m_{ij} \sin \theta_{ij} + \cos \theta_{ij}) \exp \left[ i(\mathbf{k}_i - \mathbf{k}) \cdot \mathbf{R}_j \right] \Delta x \Delta x. \]  

(16)

Similarly for s-polarisation (TE), \( R = -1 \) at both points and the term obtained is

\[ E^{(2)}_s(\theta) = -\left( \frac{2}{\pi kr} \right)^{1/2} \frac{\exp(i\varphi)}{4i} \sum_{j=1}^{N} \sum_{i=1}^{N} S'_{ji} \left[ m_j \frac{k(x_i - x_j)}{r_{ij}} - \frac{k(z_i - z_j)}{r_{ij}} \right] \]

\[ \times H_1^{(1)}(kr_{ij})k(m_{ij} \sin \theta_{ij} + \cos \theta_{ij}) \exp \left[ i(\mathbf{k}_i - \mathbf{k}) \cdot \mathbf{R}_j \right] \Delta x \Delta x. \]  

(17)

To obtain this result the following approximation was used

\[ \frac{\partial}{\partial n_i} \int_r S(x_1, z_1)S_{ji}k(m_j \sin \theta_i + \cos \theta_i)E_i(x_j, z_j)H_0^{(1)}(kr_{ij}) \, dx_j \]

\[ \approx \int_r S(x_1, z_1)S_{ji}k(m_j \sin \theta_i + \cos \theta_i)E_i(x_j, z_j) \frac{\partial H_0^{(1)}(kr_{ij})}{\partial n_i} \, dx_j, \]  

(18)

i.e. the normal derivative of the shadow function \( S_{ji} \) is ignored. The effect of this approximation on the final result is not known.

Once the field as a function of angle is known the power per unit angle at the far field is given by

\[ J(\theta) = \lim_{r \to \infty} r \left| E_s(\theta) \right|^2. \]  

(19)

For a perfect conductor \( R = 1 \) for p-polarisation and \( -1 \) for s-polarisation, therefore from (11) the modulus of the single scatter term is the same for both polarisations. The double-scatter term, however, is different for the two cases, (16) for p and (17)
for s polarisation. The sum of the single and double contributions, the ‘total’ power is obtained from

\[
J_s(\theta) = \lim_{r \to \infty} r |E_s(\theta) + E_s^{(2)}(\theta)|^2.
\] (20)

In this case the sign of the single-scatter term is very important and leads to a difference in the total scattered intensity distribution.

A test of the reliability of any method for calculating the scattered field is the unitarity or the power scattered divided by the incident power. This is given by

\[
U(\theta) = \frac{\int_0^\infty J_s(\theta') \, d\theta'}{\int_0^\infty \int_0^\pi J_\theta(\theta') \, d\theta' \, d\omega}.
\] (21)

3. Results

The equations for the single and double-scatter terms for a perfectly conducting rough surface with the parameters given in the introduction were programmed into a computer. To reduce speckle noise on the resulting solutions the angular distribution of intensity was averaged over many (typically 1000) different realisations of the surface profile all with the same statistical properties. The relative amount of energy, the unitarity, in the single, double and single + double terms is shown in the table for both polarisations. From this table it can be seen that, perhaps surprisingly, the double-scatter term decreases with increasing angle of incidence, the single scatter increases and that virtually all of the scattered energy is contained in these terms. It also appears that the rate of decrease of the amount of energy in the second-order term increases with increasing angle, for s-polarisation.

The angular distribution of the scattered light is shown in figures 3 and 4 and the results are compared with experimental curves in figures 5 and 6. Experimental results are represented by the circles. There are a number of features to note about these results. First the double-scatter term has negligible values at high scatter angles (higher than ±60°) so this is the region where the single scatter dominates, i.e. the usual single scatter physical optics approximation value is closer to the actual scattered field at these higher angles.

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<td>0.977</td>
</tr>
<tr>
<td>10°</td>
<td>0.852</td>
<td>0.142</td>
<td>0.990</td>
</tr>
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<table>
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Figure 3. The single (++) , double (***) , and single plus double (—) contributions from surface number 440 with s-polarisation incident. The backscatter angles are marked by the dashed lines to the right of the graphs. The number of frames averaged over are (a) 2000 at 0°, (b) 1000 at 10°, (c) 2000 at 20° and (d) 1000 at 40°.

Figure 4. As figure 3 but for p-polarisation incident.
Figure 5. Comparison of single plus double (—) and experimental (○ ○ ○) curves for surface number 440 with s-polarisation incident.

Figure 6. As figure 5 but with p-polarisation incident.
Secondly there is no narrow peak in the single scatter term but there is in the double-scatter one. This peak occurs at the backscatter angle and there are signs of side lobes on either side of this peak. Taking the intensity at the maximum of the peak and dividing by the intensity at the same angle of the curve obtained by simply drawing a smooth line joining the parts of the curve away from the peak gives a factor of roughly two. This is the factor predicted by the simple picture of the scattering process involving the coherent addition of waves in the backscatter direction. It is important to note that the factor of two enhancement is in the double scatter term not the total term. Again this is predicted in the simple picture which requires multiple scattering to have coherent interference for the retro-reflected light. Obviously this means that the enhancement factor in the total scatter curve will be less than two in this roughness region ($\tau/\sigma_s \approx 2.4$) as the single scatter is not zero at this angle. It may be that the factor in the total scatter term can be used to somehow give a measure of the ‘roughness’ of a surface since as the roughness is increased there will be less single scatter and more multiple scatter giving a larger enhancement.

Another point which is apparent is that since the amount of double scatter decreases and the double scatter curve has smaller values at the backscatter angle as the incidence angle is increased, the size of the peak and hence the enhancement factor decreases with increasing angles of incidence.

4. Future work

Work is progressing on extending the method to the triple-scatter or third-order term and also to the double-scatter term from dielectric surfaces. The second-order dielectric term is much more complicated than the perfect conductor since light can travel through the material and then scatter away from the surface, i.e. light can travel between grooves which would be hidden from each other considering the usual shadowing term.

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