Object fitting to the bispectral phase by using least squares

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We present a new approach for the reconstruction of two-dimensional images directly from the bispectral phase. Unlike earlier methods, the system requires no information regarding the Fourier modulus, and thus the reference star measurement is unnecessary. We give examples that use experimental binary star data and simulated data of quadruple stars.

1. INTRODUCTION

Since Labeyrie sparked off the modern era of high-angular-resolution imaging in astronomy with his proposal of speckle interferometry,1 there has been significant progress in this field. Labeyrie aimed at recovering high (spatial) frequency information regarding the object by averaging the power spectrum of a large number of short-exposure images. Reconstructing the object power spectrum up to the diffraction limit yields the high-resolution autocorrelation function of the object that provides important information if the object is simple, such as a binary star. One cannot improve on the image of a more complicated object, however, with the knowledge of its autocorrelation function. To reconstruct the object intensity, Labeyrie’s work was complemented by several techniques to retrieve the Fourier phase, e.g., Knox-Thompson,2 bispectrum,3 and phase retrieval (see, e.g., Ref. 4 for a review).

Thus it is now possible to produce diffraction-limited images routinely from short-exposure speckle data.5 However, all these techniques require the measurement not only of the object but also of a pointlike reference star to determine the appropriate transfer function.

In this paper we present a new method of reconstructing the object intensity directly from the bispectral phase. The technique of fitting the parameters of a binary star to the bispectrum phase5 has been extended to an arbitrary object. Rather than fitting the three parameters of a binary to the bispectral phase, now one adjusts all pixels in the image independently to fit the bispectral phase in a least-squares sense. Reconstruction of the object is then possible without requiring the measurement of a reference star. The new approach is akin to earlier proposals of reconstructing the object from the Fourier phase.7 For noisy speckle images, however, that method has yet to deliver consistent results.8 The advantage of the technique described here is the use of the measured bispectrum phase instead of the otherwise reconstructed object Fourier phase.

The discussion regarding the uniqueness of the solution is restricted to the influence of functions with zero Fourier phase on the reconstruction.

In Section 4 we demonstrate that the impact of this ambiguity on reconstructions from experimental and simulated data is negligible. The possibility of reconstructing an object from its bispectral phase alone is particularly appealing in conjunction with adaptive optics. Recently it was shown that postprocessing data from partially compensated images by using techniques such as bispectrum-phase reconstruction improves the image quality.9 It would be considerably advantageous in some cases (e.g., satellite imaging) if this postprocessing could be done without observing a reference star.

2. BACKGROUND

Throughout this paper lowercase letters denote functions in image space with coordinate vector \( \mathbf{x} = (x, y) \) and uppercase letters signify their Fourier transforms as a function of the spatial frequency \( \mathbf{u} = (u, v) \).

We call the true object intensity \( f(x) \) and its Fourier transform, \( F(u) \), with the Fourier phase \( \varphi(u) \). The blurred point-spread function (PSF) of the \( n \)th short-exposure image is denoted by \( h_n(x) \), and the speckle pattern is signified by \( s_n(x) \). Their respective Fourier transforms are called \( H_n(u) \) and \( S_n(u) \).

The speckle pattern of a short-exposure image is

\[
s_n(x) = f(x) \circ h_n(x), \quad n = 1, \ldots, N, \tag{1}\n\]

where the symbol \( \circ \) is used to indicate a two-dimensional convolution and \( N \) is the number of short-exposure images or frames.

The instantaneous image spectrum is the Fourier transform of Eq. (1):

\[
S_n(u) = F(u)H_n(u), \quad n = 1, \ldots, N. \tag{2}\n\]

To recover the modulus \(|F(u)|\) of the object spectrum, Labeyrie proposed to form an estimate of the power spectrum by writing

\[
\langle |S(u)|^2 \rangle = |F(u)|^2 \langle |H(u)|^2 \rangle, \tag{3}\n\]

where \( \langle \cdot \rangle \) is used to indicate the ensemble average over \( N \) short-exposure images. One corrects the speckle transfer function \( \langle |H(u)|^2 \rangle \) to obtain the true power spectrum by observing a pointlike reference star for which \(|F(u)| \) is constant and removing it from Eq. (3) by division. This procedure removes the speckle transfer function and amplifies the noise in spatial frequencies near and beyond the diffraction limit. Hence it is necessary to choose...
some form of window function, e.g., the telescope modulation transfer function (MTF), to prevent this noise from dominating the reconstruction.

The estimate of the reconstructed modulus of the object is

$$|F(u)| = \text{MTF}(u) \sqrt{\frac{\langle |S_{\text{obj}}(u)|^2 \rangle}{\langle |S_{\text{ref}}(u)|^2 \rangle}}.$$  \hspace{1cm} (4)

The difficulties in properly choosing the window function together with those arising from differences in the seeing parameters between the measurements of object and reference star suggest that it is highly desirable to avoid the reference star measurement.

The bispectrum of a real object is defined as

$$F^{(3)}(u_1, u_2) = F(u_1)F(u_2)F(-u_1 - u_2)$$

$$= |F^{(3)}(u_1, u_2)| \exp[i\psi(u_1, u_2)],$$  \hspace{1cm} (5)

where $\psi(u_1, u_2)$ is the phase of the object bispectrum. Averaging the bispectrum over all the short-exposure images yields a result that is similar to the averaged power spectrum:

$$\langle S^{(3)}(u_1, u_2) \rangle = F^{(3)}(u_1, u_2)\langle H^{(3)}(u_1, u_2) \rangle.$$  \hspace{1cm} (6)

It can be shown that $\langle H^{(3)}(u_1, u_2) \rangle$ is a function with zero phase, a property that, in the limit of an infinite number of short-exposure images, is independent of minor telescope aberrations. Hence the average bispectrum phase is a function of the object alone. It is

$$\psi(u_1, u_2) = \varphi(u_1) + \varphi(u_2) - \varphi(u_1 + u_2),$$  \hspace{1cm} (7)

where $\varphi(u)$ is the true object phase.

The problem of reconstructing the object phase is a problem of determining the set of object phases that is most consistent with the measured bispectrum phase. The first reconstructions have been performed by using recursive methods. More recently, least-squares methods have proved robust and reliable and outperform the recursive techniques, particularly in the higher spatial frequencies.

If the measurement of the bispectrum phase is denoted by $\tilde{\phi}$ and the estimate of the object phase is signified by $\hat{\varphi}$, the error sum that must be minimized reads

$$E_r = \sum \left\{ \text{mod} \, 2\pi [\tilde{\psi}_{ij} - (\hat{\varphi}_i + \hat{\varphi}_j - \varphi_{ij})]^2 W_{ij} \right\},$$  \hspace{1cm} (8)

where $i$ and $j$ are the discrete coordinates in the two-dimensional phase array and $M$ is the number of points in the bispectrum. The difference is taken modulo $2\pi$ to provide the necessary unwrapping. The weighting function $W_{ij}$ is defined as the reciprocal of the variance of the measured bispectrum phase, $\tilde{\psi}_{ij}$ which is a good measure for the signal-to-noise ratio (SNR) of $\tilde{\psi}_{ij}$ and is obtained by

$$W_{ij} = \frac{\text{Var}(\tilde{\psi}_{ij})}{\text{Var}(S_{ij})^2} = \frac{\text{Var}(S_{ij})^2}{\text{Var}_{ij}},$$  \hspace{1cm} (9)

with $\text{Var}(S_{ij})^2$ being the variance in the direction perpendicular to the direction of the complex bispectrum phasor $S_{ij}$ and $|S_{ij}|^2$ being the square of the bispectral modulus.

In practice it is not computationally feasible to use the entire bispectrum, and only those portions with a large SNR are employed. We define a subplane as the set of all points in the bispectrum that are obtained by fixing $i$ at a constant value and by varying $j$. It has been noted by Ayers et al. that those subplanes for which $|\tilde{\psi}_{ij}|$ is small usually have a significantly higher SNR. Thus, when reconstructions are quoted as being for a given number of subplanes, we use those subplanes for which $|\tilde{\psi}_{ij}|^2$ is the smallest.

Applying a standard least-squares routine such as E04DGF from the NAG library to Eq. (8) delivered consistent phase reconstructions for a variety of data. Whereas the reconstruction of the object phase from the bispectral phase is performed without a reference star, the modulus reconstruction in Eq. (4) requires measurement of the reference star. In the next paragraph we will show how the object reconstruction can be performed directly from the bispectrum phase, thus requiring no reference star.

### 3. OBJECT RECONSTRUCTION FROM THE BISPECTRAL PHASE

The relation between the object Fourier phase $\varphi(u)$ and the (real) intensity distribution $f(x)$ is given by

$$\varphi(u) = \text{atan} \left( \frac{\mathcal{F}[f(x)]}{\mathcal{F}[\bar{f}(x)]} \right).$$  \hspace{1cm} (10)

with $\mathcal{F}$ being the imaginary part and $\mathcal{F}$ being the real part of the Fourier transform of $f(x)$. The sum of Fourier phases at three different points $u_1$, $u_2$, and $u_1 + u_2$ yields the bispectral phase $\psi(u_1, u_2)$ [see Eq. (7)].

The process of fitting the intensity $f$ to the bispectral phase $\tilde{\psi}$ then consists of minimizing the error metric

$$E_r = \sum \left\{ \text{mod} \, 2\pi (\tilde{\psi}_{ij} - \hat{\psi}_{ij})^2 W_{ij} \right\},$$  \hspace{1cm} (11)

where $\tilde{\psi}$ is the measured bispectral phase and $\hat{\psi}$ is the estimated bispectral phase that is formed by the current intensity estimate $\hat{f}(k)$, with $k$ being the discrete coordinate of the two-dimensional image.

A function with zero Fourier phase cannot be reconstructed from the phase alone. More important, however, is the possibility that a solution $\hat{f}' = \hat{f}_k \odot \hat{f}_0$ instead of $\hat{f}_k$ is found, where $\hat{f}_0$ is a function with zero Fourier phase. Then $\hat{f}_k$ and $\hat{f}_0$ have the same Fourier phase and the same error sum. This question is addressed again in Section 4 when we discuss the reconstructions.

A positivity constraint for the image intensity is implemented according to a proposal by Lane by introducing an additional error sum

$$E_r = \sum _i |\hat{f}_i|^2,$$  \hspace{1cm} (12)

where $k$ is the discrete coordinate in image space and $\gamma$ is the set of points at which $\hat{f}_k$ is negative.
The sum of the error measures
\[ E = E_f + E_i \] (13)
is then minimized to fit the object to the bispectral phase.

A technique to fit the object intensity to the bispectrum was discussed by Hofmann and Weigelt. However, their method uses the complex bispectrum rather than the phase only and thus requires the measurement of a reference star for reconstruction of the bispectrum modulus.

Darling reported a method called implicit blind deconvolution. The object Fourier phase is reconstructed from the averaged bispectrum and is then used to reconstruct the object by iteration. The major difference between Darling's technique and our approach is that, in the former, one recovers the object phase explicitly before reconstructing the object intensity. In his paper on blind deconvolution Lane presents several variations of the usual deconvolution algorithms and also uses the reconstructed object phase for the blind deconvolution of a single frame. The use of the reconstructed object phase rather than the bispectral phase, the measured quantity, has the disadvantage that the measurement error cannot be considered properly. We regard it as crucial that the weighting function incorporates the SNR of the bispectral phase as in Eqs. (8) and (9).

4. RESULTS

We used a commercially available least-squares routine of the NAG library (E04DGF) to minimize the combined error sum \( E \) [Eq. (13)]. The balance between \( E_f \) and \( E_i \) depends on the starting point of the iteration. Because the absolute numbers of the intensity cannot be recovered from the Fourier phase alone, one can start with a PSF or a first estimate of the object of any absolute value. Starting with a PSF with a maximum of 100, we put a factor 10 on \( E_i \). We chose the PSF as the most general starting point.

The least-squares routine requires the first derivatives of \( E \) with respect to the image intensity \( f_k \) that are given in Appendix A. To abbreviate the computational time, it is useful to have a lookup table for the sine and cosine functions. Because every possible combination of Fourier space coordinates and image space coordinates must be stored, a 64 x 64 phase array (for a 64 x 64 image) requires 256 Mbytes of memory. If the object is known to be restricted to, e.g., a 16 x 16 array, a more manageable 16-Mbyte memory is required. We emphasize that this restriction is not imposed by the algorithm but by the limitations of currently available computer memory. We do not regard this restriction as a major hindrance.

We used experimental binary star data and quadruple star simulations (which employ a phase screen model with Kolmogorov statistics) to test the algorithm. For the simulation, the relation telescope diameter \( D \) over the Fried parameter \( r_0 \) was chosen as \( D/r_0 = 20 \). Assuming that one uses a 3.8-m telescope and a 2.2-\( \mu \)m observing wavelength, the Fried parameter \( r_0 \) is 20 cm for \( D/r_0 = 20 \). For a 2-m telescope, our simulations are for a 500-nm observing wavelength with \( r_0 = 10 \) cm for \( D/r_0 = 20 \). These values of \( r_0 \) are typical at good sites.

We used 256 frames for averaging the bispectrum. The size of the image array is 64 x 64, of which the 16 x 16 central part is always displayed. The diameter of the telescope aperture is 32 pixels. If we do not otherwise state, we use 80 subplanes of the bispectrum, which is 5% of the nonredundant bispectrum for a 64 x 64 phase array.

For comparison Fig. 1 shows the almost diffraction-limited reconstruction by using the conventional least-squares phase reconstruction [Eq. (8)] and the modulus recovery [Eq. (4)].

In Fig. 2 the raw image, i.e., the intensity distribution delivered by the new algorithm, is displayed. We restricted the number of iterations to 250 because there is no significant change in the error sum and in the intensity reconstruction after the first drop in the error sum at 50-100 iterations. (The error sum decreased by a factor of...
of only ~10 between the first estimate and the final solution.) In some cases the algorithm stopped automatically before that limit was reached because the error sum could not be further decreased within the computer accuracy. It is difficult to define a stopping criterion because any intensity distribution with zero phase does not affect the process, as discussed above in Section 3. Thus a change in the intensity estimate might cause either no change or only an extremely small change in the error sum. This ambiguity posed no problem, however, inasmuch as we never encountered a significant change in image quality with an increasing number of iterations.

There are spatial frequencies in the reconstruction that are higher than the diffraction limit because every pixel is adjusted independently from its neighbor without taking the implied convolution with the diffraction-limited PSF into consideration. This increases the spatial resolution, as can be seen from the single stars whose reconstructions are narrower than the diffraction-limited case. Figure 3 demonstrates this slight superresolving effect in frequency space. Figure 3(a) shows the perfect Fourier modulus of the simulated object, and Fig. 3(b) shows the diffraction-limited modulus. Figure 3(c) presents the modulus of the Fourier transform of the reconstruction in Fig. 2. The spectrum is reconstructed slightly beyond the diffraction limit.

However, despite the apparent benefit of a higher resolution, the overall image quality that is produced by the new algorithm is not better than for the conventional reconstruction because of the number of artifacts. That is not surprising because the additional use of the power spectra of object and reference stars provides more information. Spatial filtering with a Gaussian function to reduce the artifacts without reducing the resolution too much delivers the reconstruction displayed in Fig. 4(a). The four lowest contours mark the noise level of the reconstructions, which is at ~10%; this reconstruction should be compared with the conventional reconstruction in Fig. 1.

To test the algorithm with noisy data, we contaminated the speckle pattern with different types of noise. The first type is additive zero-mean Gaussian noise that might be typical for infrared observations. The standard deviation of the Gaussian noise is 1.25 units, with the average intensity of the speckle pattern being normalized to unity and the peak intensity being at 20 ± 5 (i.e., a peak SNR of ~20:1 but an average SNR of ~0.8).

The second type is signal-dependent noise, with the noisy speckle pattern being

\[ s_n(x) = s_n(x)[1 + g(x)] . \]

The standard deviation of the Gaussian variable \( g(x) \) is 0.25 (i.e., a SNR of ~4:1 at each point).

The reconstruction with background noise that is displayed in Fig. 4(b) is quite similar in quality to the noise-free case in Fig. 4(a). The reconstruction with signal-dependent noise in Fig. 4(c) has fewer artifacts, but the relative peak heights of the three darker components are ~30% too low compared with the central peak. This is due in part to the greater diameter of the darker components, so that the integral intensity over every component is approximately correct, as is the relative intensity among the darker components. A careful comparison between the intensity profiles of all components could improve the performance of the reconstruction in this case.

In general, the algorithm is remarkably resistant to the considerable amount of noise that we added to the data. We believe that this is due to the proper consideration of the SNR in Eq. (9).

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Fig. 4. Reconstruction of the simulated quadruple star with $D/r_0 = 20$ after spatial filtering with a Gaussian function. Compare (a) with Fig. 2, which shows the unprocessed image. In (b) Gaussian background noise has been added to the speckle frames with a peak SNR of $-20:1$. In (c) signal-dependent noise was added to the frames with a SNR of $4:1$ at each point. Forty contour lines on a linear scale are displayed. The reconstructions are remarkably unaffected by the contamination with noise.

Fig. 5. Reconstruction of another quadruple star simulation provided by Christou with $D/r_0 = 10$ and Gaussian background noise with a peak SNR of $-35:1$. (a) The conventional reconstruction with the use of a reference star. In (b) and (c) the results with the new algorithm are displayed with the use of 6 subplanes from the bispectrum for (b) and 80 subplanes for (c). Twenty lines on a linear scale are displayed. The slight superresolution and the benefits of a higher number of subplanes are readily apparent.
Fig. 6. Reconstruction of experimental binary star data ADS15267 [in (a) and (c)] and ADS15281 [in (b) and (d)]. (a) and (b), The results with the new algorithm. (c) and (d), The results of the conventional reconstructions. Using the new method provides much improved estimates of the relative peak heights.

Table 1. Relative Brightness of Binary Stars Estimated with Different Methods

<table>
<thead>
<tr>
<th>Star</th>
<th>Previous Estimate</th>
<th>New Method</th>
<th>Conventional Method</th>
</tr>
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<tbody>
<tr>
<td>ADS15267</td>
<td>0.65</td>
<td>0.62</td>
<td>0.22</td>
</tr>
<tr>
<td>ADS15281</td>
<td>0.76</td>
<td>0.65</td>
<td>0.55</td>
</tr>
</tbody>
</table>

extended object and 1000 frames for a point source, both with $D/r_0 = 10$. The data are contaminated with additive zero-mean Gaussian noise with a standard deviation of 1 when the average intensity is normalized to unity and the peak intensity is at $35 \pm 15$. Figure 5(a) displays the result of the conventional algorithm, and Figs. 5(b) and 5(c) present the reconstructions according to the new method, with 6 subplanes being used for Fig. 5(b) and 80 subplanes for Fig. 5(c). Thus increasing the number of subplanes improves the image quality. A slight superresolution can be observed again such that the small component close to the central peak is resolved.

The binary data set consists of high-light-level images that were observed through the 2.12-m telescope at the San Martir Observatory in Baja California, Mexico, at $\lambda = 516$ nm in October 1988. These data have been provided by J. Ohtsubo, who also presented a power spectrum analysis. Our reconstructed intensity distributions for ADS15267 and ADS15281 in Fig. 6 show noisier images for the reconstructions with the new algorithm [Figs. 6(a) and 6(b)] than with the conventional one [Figs. 6(c) and 6(d)]. The relative peak height was estimated previously, and the new method provides a greatly improved reconstruction of it (see Table 1). The reason is the poor modulus reconstruction that is caused by different seeing conditions for object and reference star measurement. This is one example in which difficulties in matching the seeing conditions cause the modulus reconstruction to be extremely difficult and in which the new method delivers satisfying results.

5. CONCLUSIONS

We have presented a new method for reconstructing the intensity distribution directly from the bispectral phase, which is applicable when it is not possible to obtain re-
APPENDIX A

Here we use the same notation as in Sections 2 and 3, where $f$ is the current estimate of the image intensity, $\psi$ is the bispectral phase, and $\phi$ is the object Fourier phase. The discrete coordinates in Fourier space are $i$ and $j$, and the coordinates in image space are $k$ and $l$.

The relation between $\phi$ and $f$ is [see Eq. (10)]

$$\phi = \arctan \left[ \frac{\sum f_k \sin(2\pi ik/N)}{\sum f_k \cos(2\pi ik/N)} \right], \quad (A1)$$

where $N$ is the size of the array. The summation is performed from 1 to $N$.

The bispectral phase is

$$\hat{\phi}_{ij} = \phi_i - \phi_j - \psi_{ij}. \quad (A2)$$

The error sum that must be minimized is

$$E = E_f + E_i$$

$$= \sum_{i,j} \left[ \text{mod} \left( 2\pi (\psi_{ij} - \hat{\phi}_{ij}) + f_i \right)^2 W_{ij} + \sum_j |f_i|^2 \right], \quad (A3)$$

where $\hat{\psi}$ is the measured bispectral phase. The second sum is performed over those points $\gamma$ with negative intensity [compare Eqs. (11)–(13)].

The first partial derivative of the error sum $E$ with respect to the intensity $f_k$ is

$$\frac{\partial E}{\partial f_k} = \sum_{i,j} 2 \text{mod} 2 \pi (\psi_{ij} - \hat{\phi}_{ij}) W_{ij} \frac{\partial \hat{\phi}_{ij}}{\partial f_k} + 2f_k. \quad (A4)$$

The partial derivative of $\psi_{ij}$ is the sum of the derivatives of $\phi$ [Eq. (A2)]. Because the first derivative of $\arctan(u/v)$ is $(u'/v - v'u/v^2)$ (the prime indicating the first derivative), the derivative of $\phi$ can be written as

$$\frac{\partial \phi}{\partial f_k}$$

where $R$ is the real part and $I$ is the imaginary part of the Fourier transform, being a function of the frequency coordinate $I$. The real and imaginary parts are readily available after one Fourier transforms the current estimate of the intensity distribution $f$, and the cosine and sine functions are stored in a lookup table. Grouping together these derivatives at coordinates $i$, $j$, and $i + j$ to build the derivative of the bispectral phase $\psi_{ij}$ yields the complete expression for the first derivative of the error sum from Eq. (A4).

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