Simulation of time-evolving speckle patterns using Kolmogorov statistics

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Abstract. We present a new approach for simulating time-evolving speckle patterns, by combining the shift of a rigid phase screen with its evolution. Thus, the two time-scales associated with speckle boiling and speckle motion can be adjusted independently. The statistical properties of the phase perturbation and the speckle pattern are investigated by using the temporal phase structure function and the temporal intensity correlation. We found a very good agreement with experimental results.

1. Introduction

The recent demonstration of the potential of adaptive optics systems in ground-based astronomy [1–7] has increased the efforts in this field enormously. As the temporal evolution of the turbulent atmosphere is crucial for the specification of the whole system, a computer simulation is very useful for investigating the performance [8–10]. In this paper we present an algorithm to simulate evolving speckle patterns by moving a phase screen in front of the telescope aperture. To model the speckle boiling effect the phase screen evolves, using a Markov process.

The temporal properties of speckle patterns [11–17] or the phase perturbation [18, 19] in the telescope aperture have been repeatedly measured and it has been demonstrated that a short time scale of the order of 10 milliseconds, associated with an evolution of the wavefront, and a long time-scale of the order of a second, associated with the moving wavefront, can be distinguished. These short time-scales determine how fast an adaptive optics system has to adjust to the changes in the wavefront. In speckle interferometry, the short time-scale determines the exposure time of the speckle frame in order to get the maximum signal to noise ratio. A longer exposure time would smear the speckle image and eventually lead to the long exposure image.

The models for the temporal behaviour of the turbulent atmosphere used to interpret the experiments have relied on the shift of one [12, 14] or more [20] frozen layers of turbulence. Using one moving layer gives the correct qualitative behaviour of the correlation, but requires unrealistic wind speeds of about 100 m s$^{-1}$ to give the correct short time-scale. Furthermore, this time constant does not depend on the wavelength, although this has been found experimentally to be important. With the model involving several moving layers with frozen turbulence, the short time-scale has the correct order of magnitude for realistic wind speeds and depends on the wavelength. It is probably sufficient to employ two layers so that the difference and
the sum of the single velocities determine the short and the long time-scale. The multi-layer model has the advantage that it mimics the physics of the process but it is computationally very demanding. A multi-layer model is probably essential for understanding effects of non-isoplanicity.

We have chosen a different approach to the problem by using a single evolving layer. This is very similar to the visual appearance in the telescope aperture. In the following we describe our model for evolving speckle patterns using a single layer that is evolving as well as moving. A new algorithm is used to include low frequencies with a period length much larger than the phase screen according to Kolmogorov statistics [21]. We produce a sequence of speckle images and describe them by statistical means. We show how both time constants can be adjusted, and how they affect the temporal correlation function.

2. Theory

2.1. Spatial statistics

The turbulent layer in the atmosphere with phase distribution $\phi(\mathbf{r})$ is typically described by two functions, the phase structure function $D_\phi(|\mathbf{r}|)$ and the power spectrum of the phase fluctuation $\Phi(|\mathbf{k}|)$, where $\mathbf{r}$ is the spatial coordinate in the layer and $\mathbf{k}$ is the coordinate in frequency space.

For Kolmogorov turbulence, the spatial phase structure function is [22]

$$D_{\phi,m}(|r|) = \langle (\phi(r') - \phi(r') - r)^2 \rangle = 2\langle \phi(r')^2 \rangle - \langle \phi(r') \phi(r' + r) \rangle = 6.88(|r|/r_0)^{5/3}, \quad (1)$$

where $r_0$ is the Fried parameter and $\langle \ldots \rangle$ is the ensemble average.

The power spectrum of the spatial phase fluctuation is

$$\Phi(|\mathbf{k}|) = \langle |\phi(\mathbf{k})|^2 \rangle = 0.023 r_0^{-5/3} |\mathbf{k}|^{-11/3}, \quad (2)$$

where $\hat{\phi}(\mathbf{k})$ is the Fourier transform of the phase distribution $\phi(\mathbf{r})$.

The unusual feature of Kolmogorov turbulence is that there is a correlation between two points of the turbulence however far they might be separated. In reality, the Kolmogorov model applies only when the separation of the two points lies within the limits given by the inner and outer scales of turbulence (the inertial subrange), i.e. in the range of millimetres to several metres, although the exact values of these parameters are uncertain.

The power spectrum $\Phi(|\mathbf{k}|)$ with infinite power for $|\mathbf{k}| \to 0$ reflects this characteristic, as a very low spatial frequency $|\mathbf{k}|$ corresponds to a very large distance in the atmosphere. Hence for ideal Kolmogorov turbulence, the Fourier transform relationship between the power spectrum $\Phi(|\mathbf{k}|)$ and the autocorrelation function $\langle \phi(r') \phi(r' + r) \rangle$ does not hold. The suitable expression that relates the phase structure function $D_\phi(|\mathbf{r}|)$ and the power spectrum is given by [24]

$$D_\phi(|\mathbf{r}|) = \int_{-\infty}^{+\infty} \Phi(|\mathbf{k}|) [1 - \exp(12\pi|\mathbf{k}|)|\mathbf{r}|] d\mathbf{k}. \quad (3)$$

The difficulties in simulating Kolmogorov turbulence arise from the necessary involvement of the very low frequencies and are described in detail in a previous
paper [21]. The two methods used there to simulate Kolmogorov turbulence are briefly summarized here and are extended to time evolution of the turbulent layer.

2.2. Temporal statistics

If the evolving turbulence is modelled by a frozen turbulent layer, the temporal phase structure function is simply

\[ D_{\tau, \phi}(\tau) = \langle (\phi(t) - \phi(t + \tau))^2 \rangle \]

\[ = 6.88 (\tau / \tau_0)^{5/3}, \tag{4} \]

with \( \tau_0 = r_0 / v \) the coherence time, \( r_0 \) the Fried parameter and \( v \) the wind speed. We will use this function to investigate the simulated phase screens.

In the investigation of the temporal characteristic of the speckle pattern, we employ the normalized temporal correlation function of the intensity,

\[ C(\tau) = \frac{\langle I(t)I(t+\tau) \rangle - \langle I(t) \rangle^2}{\langle I(t)^2 \rangle - \langle I(t) \rangle^2}. \tag{5} \]

The correlation time of the speckle pattern for a frozen turbulent layer is given by [23]

\[ \tau_m \approx 0.5 \frac{D}{v}, \tag{6} \]

where \( D \) is the telescope diameter and \( v \) the wind speed. Then the correlation function is proportional to the autocorrelation of the telescope aperture as a function of \( \tau \) [14], that has the shape of the diffraction limited modulation transfer function (MTF).

We obtain the above correlation time for the case of a frozen phase screen. However, if the turbulence is evolving, there is an additional, much shorter correlation time that can be adjusted by the decorrelation parameter of a Markov process.

3. Simulating phase perturbations

Usually a turbulent phase distribution is generated by creating Rayleigh distributed random numbers for the square root of the power spectrum \( \Phi(|k|) \). The mean of the random number at a frequency \( k_0 \) is set equal to the function value \( \Phi(|k|) \). The square root of the power spectrum is taken to obtain the modulus \( |\psi(k)| \) of the complex spectrum and a random phase \( \phi(k) \) with uniform distribution is added. The Fourier transform of this complex function, with the decomposition in the real and imaginary part, gives two distinct realizations of the turbulent phase distribution. This phase distribution however lacks the long distance correlation inherent in Kolmogorov turbulence, because the longest period that can be modelled is given by the reciprocal of the smallest frequency in the spectrum. Thus the slope in the phase of the telescope aperture, responsible for a shift of the centroid of the speckle image, cannot be modelled unless the array size taken for the power spectrum is many times larger than the actual telescope aperture.

This restriction can be circumvented by the addition of subharmonics as shown in [21]. The Fourier transform of a discrete array containing a sine wave with a period length longer than the size of the array has non-zero values at all frequencies. This is because the frequency of the sine wave cannot be represented by discrete values of
the spectrum as this frequency is smaller than unity. This effect can be used for
modelling the distribution in the frequency space to give a period length in object
space that is longer than the size of the array.

In practice one has to introduce a weighting function because the square of size
1 x 1 around the origin (0, 0) in the discrete array of the frequency space is now
represented by a number of samples, i.e. the number of added subharmonics, and not
only by the single sample at (0, 0). We have chosen to replace the single sample by
nine subsamples at (-1/3, -1/3), (-1/3, 0), (-1/3, 1/3) etc., each representing 1/9
of the square. Thus, each contribution has to be weighted down by 1/9. The addition
of further subsamples can be done correspondingly, i.e. subdividing the remaining
patch of size 1/9 at the origin again into nine subsamples each of size 1/81. The low
frequencies involved in this second step are (-1/9, -1/9), (-1/9, 0), (-1/9, 1/9)
etc. and the weighting factor is 1/81. This process can be repeated until the peak at
the origin of the power spectrum is sampled satisfactorily. In our calculation, we
went down to 1/3^5 requiring a weighting factor of (1/3^5)^2.

The complete spectrum, including the contribution of the subharmonics to the
frequencies on the discrete array, becomes

\[ \phi_{\text{conv}}(k, l) = (0.023)^{1/2} \left( \frac{2D}{r_0} \right)^{5/6} |(k, l)|^{-11/6} \exp \left[ i\psi(k, l) \right] \]
\[ \phi_{\text{sub}}(k, l) = (0.023)^{1/2} \left( \frac{2D}{r_0} \right)^{5/6} \text{sinc} \left[ \pi(k - k_s) \right] \text{sinc} \left[ \pi(l - l_s) \right] \]
\[ \times W(k_s, l_s) \left( k_s, l_s \right)^{-11/6} \exp \left[ i\psi(k_s, l_s) \right], \]
\[ \phi(k, l) = \phi_{\text{conv}}(k, l) + \phi_{\text{sub}}(k, l), \] (7)

where \((k, l)\) is the vector coordinate of the discrete array and \(W(k_s, l_s)\) is the weighting
factor of the added low frequency \((k_s, l_s)\). The \text{sinc} \((\cdot)\) function is defined as \text{sinc} \((x) = \sin(x)/x\). This process is described in more detail in [21] and the advantage over
the conventional method is demonstrated for static turbulence.

4. Evolving speckle fields

One method of obtaining an evolving speckle pattern is to create a very large
phase screen and move the aperture slowly over it, i.e. move a frozen turbulence in
front of the telescope. This is not very realistic and gives the right time-scale of
speckle boiling only if the velocity of the moving turbulence is extremely high.

In order to model evolving speckle fields that have both time-scales we move an
evolving turbulence in front of the telescope aperture. Thus, with the velocity of the
moving screen and the statistical evolution, two parameters can be adjusted.

To model the statistical evolution in the phase screen, the random number in its
Fourier spectrum are evolved by using a Markov process. Starting with a phase
screen created by using the procedure described in the last section, a second set of
numbers \(\phi(k, l)\) is created and they are combined with the old spectrum to obtain the
result \(\phi_{\text{new}}(k, l)\) using the procedure

\[ \phi_{\text{new}}(k, l) = \frac{\phi_{\text{old}}(k, l) + \alpha(k, l) \phi(k, l)}{[1 + \alpha^2(k, l)]^{1/2}}, \] (8)

where \(\alpha(k, l) = M2/N \|(k, l)\|\) is the decorrelation factor that affects the velocity of the
evolving process and \(N^2\) is the size of the array. We have set \(\alpha\) proportional to \|(k, l)\|,
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so that small structures related to high spatial frequencies decorrelate faster than large structures (which makes sense physically). $M$ can be set to match the short time constant of measurements.

This procedure is repeated for every new realization of a phase screen and has the feature that every new state solely depends on the old state and not on its predecessors, making it a Markov process [25].

The shift of the phase screen is performed in frequency space by adding a linear function to the phase of the complex spectrum. Then the spectrum becomes

$$\tilde{\phi}_{\text{shift}}(k, l) = \tilde{\phi}_{\text{old}}(k, l) \exp (2\pi i k N_s / N),$$

where $N_s$ is the number of pixels that the phase screen has to be shifted in the direction of the coordinate $k$ and $N^2$ is the size of the array. Since the shift is performed in Fourier space $(k, l)$, $N_s$ is not restricted to integers.

Inserting $\tilde{\phi}_{\text{shift}}(k, l)$ instead of $\tilde{\phi}_{\text{old}}(k, l)$ in equation (8) combines the Markov process with the moving screen. The short and long time-scale of the resulting sequence of speckle patterns can be adjusted by the decorrelation parameter $M$ and the phase screen shift $N_s$.

5. Results

The temporal phase structure function $D_{\tau, \phi}(\tau)$ (equation (4)) and the correlation of the intensity $C(\tau)$ (equation (5)) are used to investigate the performance of the algorithm. We have chosen a telescope aperture $D$ of 32 pixels and a Fried parameter $r_0$ of 1/10 of the telescope diameter.

5.1. Temporal phase structure function

For the investigation of $D_{\tau, \phi}(\tau)$, the windspeed is set to 64 pixels $s^{-1}$, that is 4 m s$^{-1}$ if the telescope aperture has a diameter of 2 m. 1000 frames for 1 s of observing time (i.e. 1 ms exposure time) are generated, corresponding to a shift $N_s$ of $64 / 1000 = 0.064$ pixels per frame. In order to compare our results to experimental investigations explaining the temporal correlations in the wavefront with a frozen turbulence model [18, 19] the Markov process is switched off for this part of the study. To the best of our knowledge no experiments have been performed taking the evolving character of the turbulence into consideration for measurements of the temporal correlation of the atmosphere.

If the overall shift of the phase screen, i.e. the product of number of phase screens and shift $N_s$, is greater than the size of the phase screen, the process is periodic and the temporal correlation increases for a very large temporal separation of two points. To avoid this, the size of the phase screen has to be chosen sufficiently large. In the case of a shift of 0.064 pixels/frame and a number of 1000 frames the size of the phase screen has to be at least $128 \times 128$.

Figure 1 displays the temporal phase structure function for different numbers of subharmonics as well as the ideal curve for a single frozen layer (equation (4)) with a coherence time $t_0 = 50$ ms (according to $t_0 = r_0 / v = 3.2 / 64$ s). The addition of subharmonics gives a much better fit to the ideal curve. Thus, for a phase array only four times larger than the telescope aperture the addition of subharmonics is essential for a good estimate of the temporal phase structure function.

5.2. Temporal intensity correlation

For the investigation of the intensity correlation $C(\tau)$ two wind speeds are chosen, 32 and 64 pixels $s^{-1}$, i.e. 2 and 4 m s$^{-1}$ for a 2 m aperture. The frame rate is
Figure 1. The temporal phase structure function $D_{\text{t},\phi}(\tau)$ for different numbers of additional low frequencies. The dashed line represents the theoretical result. The upper solid line displays our result if two sets of additional low frequencies are used. The lower solid line displays the phase structure function without additional low frequencies. Although the phase array used for the simulation is four times larger than the aperture the addition of subharmonics is essential for a good estimate of the temporal phase structure function.

again 1000 frames s$^{-1}$ corresponding to shifts $N_s$ of 0.032 and 0.064 pixels/frame. The decorrelation parameter $M$ is set to 0.5 and 1.0. Thus four sets of curves are generated. It is necessary to generate around 4000 frames to obtain a reasonable estimate of the long time-scale, expected to be between 250 and 500 ms. The size of the phase screen has to be at least $512 \times 512$ to avoid periodicity problems. Since the telescope aperture is a circle with 32 pixels diameter at the centre of this array, the addition of subharmonics is unnecessary in this case, as the very low spatial frequencies in the telescope aperture are modelled to a sufficient extent by using the $512 \times 512$ phase screen.

In figure 2 the temporal correlation function $C(\tau)$ (see equation (5)) for the wind speeds 32 and 64 pixels s$^{-1}$ and the decorrelation factors $M=0.5$ and 1 are displayed. In order to obtain the two time-scales we have fitted the sum of two correlation functions to the temporal correlation of the simulation. For the short time speckle boiling effect we used an exponential function and for the long time speckle motion we used the one-dimensional approximation for the diffraction limited MTF. The fitted curve reads as

$$C_{\text{fit}}(\tau) = (1-a) \exp\left(-\frac{\tau}{\tau_b}\right) + a\left(1 - \frac{\tau}{2\tau_m}\right),$$

where $a$ is a normalization factor and $\tau_b$ and $\tau_m$ are the short and the long correlation time.

The results for the short and long time-scales are given in the table. The values for the long time constants are smaller than the theoretical predictions (500 ms for $\nu=2$ m s$^{-1}$ and 250 ms for $\nu=4$ m s$^{-1}$). This could be due to the decorrelation of the phase array. It is important to note that the long time-scale depends only on the wind speed.

The exponential function for the short time correlation is the best fit to our curves. This is not in contradiction to experimental results where in some cases a
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Figure 2. The temporal correlation function $C(t)$ of the image intensity for four sets of parameters for wind speed and decorrelation. To a very good approximation the short time-scale is determined by the decorrelation factor $M$ and the long time-scale by the wind speed $v$. The correlation times are given in the table.

**Short and long time-scales of our simulations when fitting an exponential and the one-dimensional MTF to the temporal correlation function $C(t)$.**

<table>
<thead>
<tr>
<th>$M$</th>
<th>$v$</th>
<th>$\tau_b$</th>
<th>$\tau_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2 m/sec</td>
<td>47 ms</td>
<td>280 ms</td>
</tr>
<tr>
<td>1.0</td>
<td>2 m/sec</td>
<td>16 ms</td>
<td>290 ms</td>
</tr>
<tr>
<td>0.5</td>
<td>4 m/sec</td>
<td>45 ms</td>
<td>180 ms</td>
</tr>
<tr>
<td>1.0</td>
<td>4 m/sec</td>
<td>16 ms</td>
<td>190 ms</td>
</tr>
</tbody>
</table>

Gaussian and in other cases an exponential curve are suitable for the short time correlation [11–16]. The short time constants $\tau_b$ that we obtain are between 16 and 47 ms and are adjusted by the decorrelation factor $M$ alone.

### 6. Conclusions

We have presented a novel approach for the simulation of time-evolving speckle patterns. The simulation of the temporal phase structure function of the wavefront and the temporal intensity correlation of the image agrees well with experiments and theory. Modelling the evolving and moving atmosphere by applying a Markov process to a moving phase screen permits the adjustment of the two time-scales in the evolving speckle pattern independently.
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References