Elementary functions: propagation of partially coherent light

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The theory of propagation of partially coherent light is well known, but performing numerical calculations still presents a difficulty because of the dimensionality of the problem. We propose using a recently introduced method based on the use of elementary functions [Wald *et al.*, Proc. SPIE **6040**, 59621G (2005)] to reduce the integrals to two dimensions. We formalize the method, describe its inherent assumptions and approximations, and introduce a sampling criterion for adequate interpolation. We present an analysis of some special cases, such as the Gaussian Schell-model beam, and briefly discuss generalized numerical propagation of two-dimensional field distributions. © 2009 Optical Society of America

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1. INTRODUCTION

The theory of propagation of partially coherent light is well known [1–3], basically taking the form of diffraction integrals that are extensions of those used for coherent or incoherent light. When it comes to performing these integrations numerically, a problem arises: the integrals are four-dimensional (4D) rather than two-dimensional (2D) as for purely coherent or incoherent light. The reason is that for partially coherent light, the second-order correlation between fields at different points in space must be taken into account, so integration must be performed not just over all points of a 2D field distribution but over all pairs of points. The quantity representing the field correlations can be either the mutual coherence function (time domain) or the cross-spectral density (frequency domain). Numerical 4D integration presents a problem even with today's vastly improved computers. One aspect is speed; propagation of partially coherent light is relatively time-consuming. The other and more serious aspect is computer memory capacity: assuming, for example, a sampling of 500 points along each axis (relatively modest sampling by modern standards), $500^4 = 62.5 \times 10^9$ points of data, or 250 gigabytes of fast memory at single precision (where 4 bytes are used to represent each number).

There are methods of avoiding these extended calculations, all with their distinct advantages and disadvantages. Hopkins's method [4] applies to a perfectly incoherent primary source: the image intensity for each source point is computed and summed over all mutually uncorrelated source points. The Fourier transform method [5] is helpful if the degree of coherence remains the same in each simulation, but a change in the degree of coherence requires new calculations before performing the propagation. The coherent-mode decomposition [3] has obvious advantages, in that it reduces the 4D integral to a 2D one, which has to be performed once for each mode: when the degree of coherence is high, the number of modes is small. The main disadvantage is that the modes are particular to a specific field; i.e., if the intensity or coherence distribution changes, one needs to recalculate the modes. The modes are found from a 2D integral equation that has analytical solutions only in special circumstances and are time-consuming to calculate numerically. Another method is based on communication modes [6], which have recently been demonstrated for partially coherent fields in free propagation [7]. The main advantage is that the modes are specific to the system and can be used for any field. The disadvantages are difficulties in finding the modes for any other system than free-space propagation and that the number of modes tends to be large. The method is best suited for relatively low degrees of coherence, where the number of modes in field-specific expansions is even higher. A radiometric (phase-space) approach, first suggested by Wolf [8], is applicable to quasi-homogeneous sources [9–11] and is related to the use of Wigner functions [12]. Another recent method is generation of multiple speckled fields that are propagated separately and then their intensities co-added [13,14]. The method resembles the real physical situation and is flexible since it can be used for many different kinds of coherence distributions, but the number of propagated fields has to be large ($\approx$10,000) for good accuracy.

In this paper we apply a method recently suggested by Wald *et al.* [15], which is similar to the coherent-mode method but easier to handle numerically. Using what amounts to approximate modes (which we will refer to as *elementary functions* using the terminology of Gabor) that are easily found analytically or numerically, propagation can be reduced to performing a series of 2D integrations. Unlike the coherent-mode expansion, the elementary function method is not mathematically exact, and we show that certain approximations limit its application to relatively well-behaved fields. Such fields could, for example, be produced by the short-wavelength partially co-

herent excimer lasers used in lithography. An elementary function expansion has also been applied in both the space–time and space–frequency domains [16–19].

The theory of the expansion is outlined in Section 2, starting with the continuous version of the exact transform in Subsection 2.A. This part follows the approach presented by Wald *et al.* applying their original transform, with clarifications and additions to the approximations and assumptions and their implications. In Subsection 2.B we introduce the discrete version of the theory, which is an expansion in a discrete set of elementary functions resembling a Gabor expansion [20]. Special attention is paid to the sampling and its implications for the result. In Section 3, three examples are analytically analyzed: the fully coherent case, the incoherent case, and the Gaussian Schell-model beam. Section 4 briefly discusses numerical aspects, and Section 5 contains discussion and conclusions.

2. THEORY OF THE EXPANSION

The transform that forms the basis of the theoretical approach was introduced in 2005 by Wald *et al.* [15] In Subsection 2.A we follow their derivation, with some clarifications and additions to the conditions of applicability, which shows that the transform yields an exact version of the cross-spectral density. However, in order to use it numerically, we need to turn it into a discrete sum, or an expansion rather than a transform. This approach, which is mentioned in [15] but not proved, is outlined in Subsection 2.B. We also develop a sampling criterion to calculate how densely the basis functions need to be placed, and show that the sampling density affects the values of the expansion coefficients.

The transform has similarities to the complex spectrogram [21], while the expansion in some ways resembles the Gabor signal expansion [20] which has also been realized for partially coherent light [22]. Gabor's signal expansion was shown by Bastiaans [21] to be the discrete version of the complex spectrogram if properly sampled.

A. Continuous Case

Let us assume that we are dealing with a partially coherent field, described by the cross-spectral density $G_0(\mathbf{r}_1,\mathbf{r}_2)$. The intensity is $I_0(\mathbf{r})=G_0(\mathbf{r},\mathbf{r})$. All quantities are implicitly assumed to depend on the frequency ν of the light, e.g., $G_0(\mathbf{r}_1,\mathbf{r}_2,\nu)$ and $I_0(\mathbf{r},\nu)$, but the notation is dropped for the rest of this discussion. It is assumed that we know $G_0(\mathbf{r}_1,\mathbf{r}_2)$ over a plane, two-dimensional region, where $\mathbf{r}_j=(x_j,y_j)$, and we wish to propagate it to any other region. We also need to make an important assumption: that the cross-spectral density is real in the plane where we start the propagation. For example, this implies that the beam must not be diverging or converging since this adds complex phase terms to the cross-spectral density. These fields can still be considered if the diverging or converging part are included in the propagation and not regarded as parts of the cross-spectral density. The propagation is performed using

$$G(\mathbf{r}_1,\mathbf{r}_2)=\int\int_{\infty} \mathrm{d}^2 r_1' \int\int_{\infty} \mathrm{d}^2 r_2' h^*(\mathbf{r}_1,\mathbf{r}_1') h(\mathbf{r}_2,\mathbf{r}_2') G_0(\mathbf{r}_1',\mathbf{r}_2'), \tag{1}$$

where $h(\mathbf{r},\mathbf{r}')$ is an impulse response function and depends on the system. It could, for example, represent propagation in free space or through an imaging system.

Introducing the transform, we assume that the initial cross-spectral density may be written as [15]

$$G_0(\mathbf{r}_1,\mathbf{r}_2)=\int\int_{\infty} \mathrm{d}^2 r' a(\mathbf{r}') f(\mathbf{r}_1-\mathbf{r}') f(\mathbf{r}_2-\mathbf{r}'), \tag{2}$$

where the elementary function $f(\mathbf{r})$ is real and even. This differs from the expression used from Wald *et al.* [15] in that we use a general coefficient $a(\mathbf{r}')$, while they force positivity by using $a^2(\mathbf{r}')$. We have also generalized it to two dimensions. The expression describes a transform, and provided that the functions $a(\mathbf{r}')$ and $f(\mathbf{r})$ exist, it is exact.

Wald *et al.* [15] introduce a method for finding $a(\mathbf{r}')$ and $f(\mathbf{r})$, thus proving that the transform in Eq. (2) exists for many well-behaved beams. Following their analysis, we take the Fourier transform of the cross-spectral density,

$$\begin{aligned}\hat{G}_0(\mathbf{u}_1,\mathbf{u}_2)=&\int\int_{\infty} \mathrm{d}^2 r_1 \int\int_{\infty} \mathrm{d}^2 r_2 \\ &\times \int\int_{\infty} \mathrm{d}^2 r' a(\mathbf{r}') f(\mathbf{r}_1-\mathbf{r}') f(\mathbf{r}_2-\mathbf{r}') \\ &\times \exp(-2\pi \mathrm{i}\mathbf{u}_1\cdot\mathbf{r}_1)\exp(-2\pi \mathrm{i}\mathbf{u}_2\cdot\mathbf{r}_2),\end{aligned} \tag{3}$$

where the spatial frequencies are $\mathbf{u}_j=(u_j,v_j)$. The Fourier transform of a function $f(\mathbf{r})$ and its inverse are defined according to

$$\hat{f}(\mathbf{u})=\int\int_{-\infty}^{\infty} \mathrm{d}^2 r f(\mathbf{r})\exp(-2\pi \mathrm{i}\mathbf{u}\cdot\mathbf{r}) \tag{4}$$

and

$$f(\mathbf{r})=\int\int_{-\infty}^{\infty} \mathrm{d}^2 u \hat{f}(\mathbf{u})\exp(2\pi \mathrm{i}\mathbf{r}\cdot\mathbf{u}). \tag{5}$$

Changing the order of integration in Eq. (3) and applying the shift theorem twice yields

$$\begin{aligned}\hat{G}_0(\mathbf{u}_1,\mathbf{u}_2)=&\hat{f}(\mathbf{u}_1)\hat{f}(\mathbf{u}_2)\int\int_{\infty} \mathrm{d}^2 r' a(\mathbf{r}') \\ &\times \exp(-2\pi \mathrm{i}(\mathbf{u}_1+\mathbf{u}_2)\cdot\mathbf{r}'),\end{aligned} \tag{6}$$

where $\hat{f}(\mathbf{u})$ is the two-dimensional Fourier transform of $f(\mathbf{r})$. Since we have assumed that $f(\mathbf{r})$ is real and even, so is its Fourier transform $\hat{f}(\mathbf{u})$, i.e., $\hat{f}(\mathbf{u})=\hat{f}(-\mathbf{u})$. Consequently, we can evaluate

$$\hat{G}_0(\mathbf{u},-\mathbf{u})=C[\hat{f}(\mathbf{u})]^2 \tag{7}$$

provided that the integral

$$C = \int \int_{\infty} \mathrm{d}^2 r' a(\mathbf{r}') \tag{8}$$

exists. This condition will be considered later on. The actual value of the constant is of no real interest, since we now know that

$$\hat{f}(\mathbf{u}) \propto \sqrt{\hat{G}_0(\mathbf{u},-\mathbf{u})} \tag{9}$$

and the shape of the function $f(\mathbf{r})$ can be found from the inverse Fourier transform. It can then be normalized.

Next, the coefficient function $a(\mathbf{r}')$ must be found. We know that the intensity may be written

$$I_0(\mathbf{r}) = G_0(\mathbf{r},\mathbf{r}) = \int \int_{\infty} \mathrm{d}^2 r' a(\mathbf{r}') f^2(\mathbf{r}-\mathbf{r}'). \tag{10}$$

Finding its Fourier transform yields, after a change of integration order and application of the shift theorem,

$$\hat{I}_0(\mathbf{u}) = \hat{a}(\mathbf{u})\widehat{f^2}(\mathbf{u}), \tag{11}$$

where $\hat{I}_0(\mathbf{u})$ is the Fourier transform of $I_0(\mathbf{r})$, $\hat{a}(\mathbf{u})$ the Fourier transform of $a(\mathbf{r})$, and $\widehat{f^2}(\mathbf{u})$ the Fourier transform of $f^2(\mathbf{r})$, the square of $f(\mathbf{r})$. So

$$\hat{a}(\mathbf{u}) = \frac{\hat{I}_0(\mathbf{u})}{\widehat{f^2}(\mathbf{u})}, \tag{12}$$

provided that $\widehat{f^2}(\mathbf{u}) \neq 0$ for all values of $\mathbf{u}$. The coefficient function $a(\mathbf{r})$ can then be found by an inverse Fourier transform.

We earlier saw that this analysis holds if the constant C in Eq. (8) exists. Now we can test this assumption by calculating

$$C = \int \int_{\infty} \mathrm{d}^2 r' \int \int_{\infty} \mathrm{d}^2 u \frac{\hat{I}_0(\mathbf{u})}{\widehat{f^2}(\mathbf{u})} \exp 2\pi i \mathbf{u}\cdot\mathbf{r}'. \tag{13}$$

Changing the order of integration allows us to evaluate the integral in $\mathbf{r}'$ as a 2D Dirac delta function $\delta(u)\delta(v)$ and evaluate the integral in $\mathbf{u}$ to give

$$C = \frac{\hat{I}_0(\mathbf{0})}{\widehat{f^2}(\mathbf{0})}. \tag{14}$$

This value exists as $\widehat{f^2}(0) \neq 0$ by definition.

Thus it has been proved that the functions $a(\mathbf{r}')$ and $f(\mathbf{r})$ exist and consequently that the transform in Eq. (2) is valid. In addition to the initial assumptions, we were forced to assume that $\widehat{f^2}(\mathbf{u})$ is nonzero for all $\mathbf{u}$. This assumption clearly limits the method to "well-behaved" cross-spectral densities $G_0(\mathbf{r}_1,\mathbf{r}_2)$ with no negative values and no sharp edges. For example, the method would work for Gaussians or super-Gaussians but not for step functions. Most fields of interest will come from low-coherence lasers or possibly LEDs and are likely to fall within these criteria.

Propagation is performed by insertion of Eq. (2) into Eq. (1). Rearranging the integration order yields

$$G(\mathbf{r}_1,\mathbf{r}_2) = \int \int_{\infty} \mathrm{d}^2 r' a(\mathbf{r}') \int \int_{\infty} \mathrm{d}^2 r_1' h^*(\mathbf{r}_1,\mathbf{r}_1') f(\mathbf{r}_1'-\mathbf{r}') \times \int \int_{\infty} \mathrm{d}^2 r_2' h(\mathbf{r}_2,\mathbf{r}_2') f(\mathbf{r}_2'-\mathbf{r}'). \tag{15}$$

Writing part of the integral as

$$f_{\mathrm{prop}}(\mathbf{r},\mathbf{r}') = \int \int_{\infty} \mathrm{d}^2 r'' h(\mathbf{r},\mathbf{r}'') f(\mathbf{r}''-\mathbf{r}'), \tag{16}$$

representing the propagated elementary function, we can write the propagated cross-spectral density as

$$G(\mathbf{r}_1,\mathbf{r}_2) = \int \int_{\infty} \mathrm{d}^2 r' a(\mathbf{r}') f^*_{\mathrm{prop}}(\mathbf{r}_1,\mathbf{r}') f_{\mathrm{prop}}(\mathbf{r}_2,\mathbf{r}'). \tag{17}$$

This is no real gain in computation speed or memory demands, since the integral has the same dimensionality as Eq. (1). But if the light has a relatively high degree of coherence, it is enough to sample Eq. (2) at a rather small number of discrete points in $\mathbf{r}'$. This decreases the dimensionality. This approximation will be treated in the next section.

To reach the results above, three assumptions were made. First, the initial cross-spectral density was assumed to be real. This puts limitations on the applicability, but it could be fulfilled for most beams. Divergent or convergent beams will have complex cross-spectral densities but can be modeled as part of the impulse response function h instead. Second, it was assumed that the elementary function $f(\mathbf{r})$ is real and even. For real and even cross-spectral densities this should not cause any problems. Third, it was assumed that $\widehat{f^2}(\mathbf{u})$ has no zeros. This may be the most restricting assumption, limiting the method to cross-spectral densities made up of smooth envelope functions such as Gaussians or super-Gaussians.

B. Discrete Expansion

The discrete case was suggested by Wald *et al.* [15] but not elucidated. Instead of the exact transform in Eq. (2) we use the corresponding discrete sum, namely, the expansion

$$G_0(\mathbf{r}_1,\mathbf{r}_2) = \sum_n \sum_m a_{mn} f(\mathbf{r}_1-\mathbf{r}_{mn}) f(\mathbf{r}_2-\mathbf{r}_{mn}). \tag{18}$$

Here $\mathbf{r}_{mn}=(x_m,y_n)$ are sampled on a grid as $x_m=m\Delta x$ and $y_n=n\Delta y$. All other assumptions are the same as in the continuous section. The values of Δx and Δy will be found by establishing a sampling criterion. We will prove that Eq. (18) is valid by finding functions $f(\mathbf{r})$ and sampling coefficients a_{mn} to fulfill the relation, in a manner resembling the continuous approach.

First, we take the Fourier transform of the cross-spectral density to get

$$\hat{G}_0(\mathbf{u}_1,\mathbf{u}_2) = \int\int_{\infty} \mathrm{d}^2 r_1 \int\int_{\infty} \mathrm{d}^2 r_2 \sum_m \sum_n a_{mn} f(\mathbf{r}_1 - \mathbf{r}_{mn}) \times f(\mathbf{r}_2 - \mathbf{r}_{mn}) \exp(-2\pi \mathrm{i}\mathbf{u}_1 \cdot \mathbf{r}_1) \exp(-2\pi \mathrm{i}\mathbf{u}_2 \cdot \mathbf{r}_2). \tag{19}$$

Interchanging the order of integrations and sum, and performing the Fourier transforms using the shift theorem, yields

$$\hat{G}_0(\mathbf{u}_1,\mathbf{u}_2) = \hat{f}(\mathbf{u}_1)\hat{f}(\mathbf{u}_2) \sum_m \sum_n a_{mn} \exp(-2\pi \mathrm{i}(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{r}_{mn}). \tag{20}$$

Again, real and even $f(\mathbf{r})$ leads to

$$\hat{G}_0(\mathbf{u},-\mathbf{u}) = \hat{f}^2(\mathbf{u}) \sum_m \sum_n a_{mn} \tag{21}$$

and, provided that the sum exists, to

$$\hat{f}(\mathbf{u}) \propto \sqrt{\hat{G}_0(\mathbf{u},-\mathbf{u})}. \tag{22}$$

Thus we have found the elementary functions and proved that the expansion does exist. The elementary functions will not necessarily be orthogonal. In fact, with most distributions we can assume they are not. For nonorthogonal sets of functions, finding the expansion coefficients becomes much more difficult than for orthogonal sets of functions. We can regard Eq. (18) as an interpolation function, where the coefficients a_{mn} are some kinds of samples of the cross-spectral density. This allows us to use the sampling and interpolation theory developed for nonorthogonal basis sets [23], which can be used both to find a sampling criterion and to retrieve the values of the coefficients. The traditional Shannon–Whittaker sampling criterion cannot be applied, since it implies that sinc functions of appropriate width will be used as interpolation functions. In our case the interpolating function, i.e., the elementary function, will not be a sinc function, and its width will be quite different from that assumed in traditional sampling and interpolation.

Sampling and interpolation can be investigated using the intensity rather than the cross-spectral density. The intensity can be written as

$$I_0(\mathbf{r}) = G_0(\mathbf{r},\mathbf{r}) = \sum_m \sum_n a_{mn} f^2(\mathbf{r} - \mathbf{r}_{mn}), \tag{23}$$

where $\varphi_{mn}(\tilde{\mathbf{r}}) = Df^2(\Delta x(\tilde{x}-m), \Delta y(\tilde{y}-n))$ can be regarded as our basis functions. The new coordinates $\tilde{\mathbf{r}} = (\tilde{x},\tilde{y}) = (x/\Delta x, y/\Delta y)$ ensure that the function is sampled at integer values, and the constant D ensures that $\hat{\varphi}(0) = 1$. These small changes allow us to use the three conditions for expansion as presented by Unser [23]. First, the sequence of coefficients must be square-integrable, that is,

$$\|a\|^2 = \sum_m \sum_n |a_{mn}|^2 \tag{24}$$

must exist. In Appendix A, it is shown to exist for any of the cases we consider.

Second, the set of basis functions needs to form a Riesz basis. That means that the condition

$$A \cdot \|a\|^2 \leq \left\| \sum_m \sum_n a_{mn} \varphi_{mn}(\mathbf{r}) \right\|^2 \leq B \cdot \|a\|^2, \tag{25}$$

where $A > 0$ and $B < \infty$, must be fulfilled. The first inequality implies that the functions are linearly independent, while the second is fulfilled for any physical elementary functions (see Appendix A). The linear independence needs to be checked for each new set of functions.

The third condition is by far the most demanding and is also the one that will be used to obtain a sampling criterion. It is known as the partition of unity condition:

$$\sum_n \sum_m \varphi(\tilde{x}+m, \tilde{y}+n) = 1, \tag{26}$$

for all real $(\tilde{x},\tilde{y})$. In the Fourier domain it translates as

$$\hat{\varphi}(m,n) = \delta_m \delta_n, \tag{27}$$

where the difference from [23] originates from different definitions of the Fourier transform and

$$\delta_m = \begin{cases} 1 & : \quad m = 0 \\ 0 & : \quad m \neq 0 \end{cases}, \tag{28}$$

where m is an integer. This condition affects how closely a function can be reproduced, by making the sampling step sufficiently small. This is also the most problematic condition, since our elementary functions are not likely to fulfill it exactly. It will be fulfilled by, e.g., sinc functions whose Fourier transforms are rectangular functions or by B-splines whose Fourier transforms are powers of sinc functions with zeros fulfilling Eq. (27) [23]. But our elementary functions are likely to be some kind of smooth envelope functions, for example Gaussians. (These are the elementary functions for Gaussian Schell-model beams, as shown in Section 3.) Their Fourier transforms will be similar, declining fairly rapidly and strictly approaching zero only as the coordinate goes to infinity. So we are forced to accept an approximate version of the partition of unity condition. This removes the formal guarantee that the expansion will be accurate to a certain limit, so care must always be taken to ensure that the expansion is numerically adequate. Nonetheless it gives a good idea of the sampling required if we use the condition that

$$\hat{\varphi}(\mathbf{0}) = 1, \tag{29}$$

while

$$\hat{\varphi}(1,1) < c, \tag{30}$$

where $c \leq 0$ is a small value considered negligible in the context and we have assumed that the function is positive with a single maximum at $(\tilde{x},\tilde{y}) = (0,0)$. Changing notation from $\varphi(\tilde{\mathbf{r}})$ to $f^2(\mathbf{r})$ changes the conditions into

$$\hat{f}^2(\mathbf{0}) = 1/D \tag{31}$$

and

$$\hat{f}^2\left(\frac{1}{\Delta x}, \frac{1}{\Delta y}\right) < c/D. \tag{32}$$

Equation (32) is then used to determine the sampling intervals Δx and Δy. If the values of Δx or Δy are very sensitive to changes in c—for example, if $\Delta x \propto c$—it is an in-

dication the elementary function does not form a good basis for the expansion. However, for elementary functions whose Fourier transforms have rapidly declining tails, which is likely to occur for physical functions, the approximation is valid and the elementary function suitable for the expansion. For example, as we shall see in Subsection 3.C, for Gaussian elementary functions $\Delta x \propto 1/\sqrt{\ln(1/c)}$. Here c can vary considerably with only minor changes to Δx, and this stability shows that the elementary function is suitable.

Once the sampling has been established, we aim to find the coefficients a_{mn}. The formal approach, which works for most elementary functions, is to find a set of analyzing functions, or window functions, that are biorthogonal to φ_{mn} [21,23]. Since this is a fairly complicated procedure, whether done analytically or numerically, we opt instead for a simplified verison that will give us the coefficients for a subclass of $f^2(\mathbf{r}-\mathbf{r}_{mn})$ that fulfil $\hat{f}^2(\mathbf{u}) \neq 0 \ \forall \mathbf{u}$. We start by taking the Fourier transform of the intensity in Eq. (23), which yields

$$\hat{I}_0(\mathbf{u}) = \hat{f}^2(\mathbf{u}) \sum_m \sum_n a_{mn} \exp(-2\pi i \mathbf{u}\cdot\mathbf{r}_{mn}). \tag{33}$$

Since $\hat{f}^2(\mathbf{u}) \neq 0$ for all $\mathbf{u}$ we find that

$$\frac{\hat{I}_0(\mathbf{u})}{\hat{f}^2(\mathbf{u})} = \sum_m \sum_n a_{mn} \exp(-2\pi i u m \Delta x)\exp(-2\pi i v n \Delta y). \tag{34}$$

This is an expansion of a known function into complex exponentials that is valid over $(-\infty,\infty)\cdot(-\infty,\infty)$. For these basis functions it is easy to find the analyzing functions

$$w_{kl}(u,v) = \Delta x \Delta y \ \mathrm{rect}\left(\frac{u}{1/2\Delta x}\right)\mathrm{rect}\left(\frac{v}{1/2\Delta y}\right) \times \exp(-2\pi i k \Delta x u)\exp(-2\pi i l \Delta y v), \tag{35}$$

where the rectangular function is one for $|u| \leq 1/2\Delta x$ and $|v| \leq 1/2\Delta y$ and zero otherwise. This term explains why these functions are sometimes referred to as window functions: it selects a window on u. It is readily shown that these functions are biorthogonal to the exponential functions in Eq. (34), i.e., that they fulfill the relation

$$\int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \exp(-2\pi i m \Delta x u)\exp(-2\pi i n \Delta y v) w_{kl}^*(u,v) = \delta_{m-k}\delta_{n-l}. \tag{36}$$

Multiplying both sides of Eq. (34) by $w_{kl}^*(u,v)$ and integrating over u yields, using the biorthogonality property in Eq. (36), the coefficient

$$a_{mn} = \Delta x \Delta y \int_{-1/2\Delta x}^{1/2\Delta x} du \int_{-1/2\Delta y}^{1/2\Delta y} dv \frac{\hat{I}_0(u,v)}{\hat{f}^2(u,v)} \times \exp(2\pi i (m\Delta x u + n \Delta y v)). \tag{37}$$

From this expression it becomes clear that the values of the coefficients are affected by the sampling density. For example, consider a_{00} for two different samplings: if the elementary functions are close together, a_{10} and a_{-10} will contribute to the value at $(x,y)=(0,0)$ and make a_{00} smaller than its value for a larger sampling interval Δx.

Finally, we have to check that the sum in Eq. (21) exists. Inserting the values of a_{mn} and interchanging the order of sums and integrals yields

$$\sum_m \sum_n a_{mn} = \int_{-1/2\Delta x}^{1/2\Delta x} du \int_{-1/2\Delta y}^{1/2\Delta y} dv \frac{\hat{I}_0(u,v)}{\hat{f}^2(u,v)} \times \sum_m \exp(2\pi i m \Delta x u) \sum_n \exp(2\pi i n \Delta y v). \tag{38}$$

Since the sums over exponential terms are the Fourier series of two delta functions $\delta(u)/\Delta x$ and $\delta(v)/\Delta y$ on $(-1/\Delta x, 1/\Delta x)$ and $(-1/\Delta y, 1/\Delta y)$, we finally get

$$\sum_m \sum_n a_{mn} = \Delta x \Delta y \frac{\hat{I}_0(\mathbf{0})}{\hat{f}^2(\mathbf{0})}, \tag{39}$$

which exists since we have already assumed $\hat{f}^2(\mathbf{u}) \neq 0$ for all relevant $\mathbf{u}$. Thus we have found both the elementary functions and the expansion coefficients of Eq. (18) and proved that the expansion exists.

In this description, propagation is given by

$$G(\mathbf{r}_1,\mathbf{r}_2) = \sum_m \sum_n a_{mn} \int\int_{\infty} d^2 r_1' h^*(\mathbf{r}_1,\mathbf{r}_1') f(\mathbf{r}_1' - \mathbf{r}_{mn}) \times \int\int_{\infty} d^2 r_2' h(\mathbf{r}_2,\mathbf{r}_2') f(\mathbf{r}_2' - \mathbf{r}_m n) \tag{40}$$

or

$$G(\mathbf{r}_1,\mathbf{r}_2) = \sum_m \sum_n a_{mn} f_{\mathrm{prop}}^*(\mathbf{r}_1,\mathbf{r}_{mn}) f_{\mathrm{prop}}(\mathbf{r}_2,\mathbf{r}_{mn}), \tag{41}$$

where $f_{\mathrm{prop}}(\mathbf{r},\mathbf{r}_{mn})$ is the propagated mode mn as given by

$$f_{\mathrm{prop}}(\mathbf{r},\mathbf{r}_{mn}) = \int\int_{\infty} d^2 r' h(\mathbf{r},\mathbf{r}') f(\mathbf{r}' - \mathbf{r}_{mn}). \tag{42}$$

The propagation has been reduced to 2D integrals used to propagate the elementary functions. Since the number of modes will be small if the degree of coherence is relatively high, numerical propagation will be swift compared to using the diffraction integrals directly, and of course the fast memory requirements are dramatically reduced.

Summarizing, all the assumptions for the continuous theory still apply (see end of Subsection 2.A), but there are three additional assumptions relating to the conditions placed on the basis functions. First, the elementary function must be square-integrable on $(-\infty,\infty)$, a condition that will be fulfilled for any physical square-integrable intensity distribution (see Appendix A) and consequently will cause no problems. Second, the basis should be linearly independent, which again will be fulfilled by any physical version of shifted elementary functions. Third, and most important, the elementary function must fulfill the partition of unity condition in Eq. (27), either exactly or to a good degree of approximation. If the elementary function is a smooth envelope function such as, e.g., a

Gaussian, this is accomplished by fulfilling the sampling condition in Eqs. (31) and (32). This condition is not likely to add any more restrictions than the earlier assumption that $\hat{f}^2(\mathbf{u})$ has no zeros.

3. ANALYTICAL EXPANSIONS FOR SOME EXAMPLE FIELDS

Some examples that can be solved analytically are presented to demonstrate the method. The first two, fully coherent and incoherent light, are trivial but illuminate the principles of the method. The third case, of Gaussian Schell-model beams, presents a situation likely to appear in real beams and clearly demonstrates the possibilities of the method. All the examples are presented for (1D) fields to facilitate the presentation, but all the results apply equally well to 2D fields.

A. Fully Coherent Light

For fully coherent light, the cross-spectral density may be written as

$$G_0(x_1,x_2) = \sqrt{I_0(x_1)}\sqrt{I_0(x_2)}, \tag{43}$$

where $I_0(x)$ is the intensity. It is separable in x_1 and x_2, so its Fourier transform may be written as the product

$$\hat{G}_0(u_1,u_2) = \mathcal{F}\{\sqrt{I_0(x_1)}\}(u_1)\cdot\mathcal{F}\{\sqrt{I_0(x_2)}\}(u_2), \tag{44}$$

where $\mathcal{F}$ represents the Fourier transform. Evaluating $G(u,-u)$ quickly yields

$$\hat{f}(u) \propto \mathcal{F}\{\sqrt{I_0(x)}\}(u) \tag{45}$$

and consequently

$$f(x) \propto \sqrt{I_0(x)}. \tag{46}$$

This confirms what was obvious already from Eq. (43): the cross-spectral density is separable and thus already written on the form of Eq. (18) with $f(x)=\sqrt{I_0(x)}$, $a_0=1$, and $a_m=0$ for $m\neq 0$. Only one mode is required.

B. Incoherent Light

The cross-spectral density for incoherent light is

$$G_0(x_1,x_2) = \sqrt{I_0(x_1)}\sqrt{I_0(x_2)}\delta(x_1-x_2), \tag{47}$$

and its Fourier transform is

$$\hat{G}_0(u_1,u_2) = \hat{I}_0(u_1+u_2). \tag{48}$$

So $G(u,-u)=\hat{I}_0(0)$ is a constant, and $\hat{f}(u)$ must be a constant too. Then $f(x)\propto\delta(x)$, and since a delta function is used for interpolation the sampling interval Δx must be infinitely small and the continuous theory must be used. The coefficients a_m can be found from Eq. (12), but since the continuous theory is used, there is no gain in computational speed. Besides, for incoherent light the normal method of propagation reduces to solving 2D integrals, so the modal method is not needed from a numerical point of view.

C. Gaussian Schell-Model Beams

The cross-spectral density of a Gaussian Schell-model beam is given by

$$G_0(x_1,x_2) = A\exp\left(-\frac{x_1^2}{2\sigma_\mathrm{I}^2}\right)\exp\left(-\frac{x_2^2}{2\sigma_\mathrm{I}^2}\right)\exp\left[-\frac{(x_1-x_2)^2}{\sigma_\mathrm{g}^2}\right], \tag{49}$$

where σ_I represents the width of the intensity distribution and σ_g the width of the coherence distribution. Making the substitution

$$\frac{1}{a^2} = \frac{1}{2\sigma_\mathrm{I}^2} + \frac{1}{\sigma_\mathrm{g}^2}, \tag{50}$$

the 2D Fourier transform of the cross-spectral density is found to be

$$\hat{G}_0(u_1,u_2) = A\pi\frac{a^2}{\sqrt{1-a^4/\sigma_\mathrm{g}^4}} \times\exp\left[-\pi^2\left(u_1^2+u_2^2+2u_1u_2\frac{a^2}{\sigma_\mathrm{g}^2}\right)\frac{a^2}{1-a^4/\sigma_\mathrm{g}^4}\right]. \tag{51}$$

The calculation is tedious but not complicated and requires completing the square of the exponent and then evaluating the Fourier transforms of Gaussian functions. Evaluating at $(-u,u)$ yields

$$\hat{G}_0(u,-u) = A\pi\frac{a^2}{\sqrt{1-a^4/\sigma_\mathrm{g}^4}}\exp\left(-2\pi^2u^2\frac{a^2}{1+a^2/\sigma_\mathrm{g}^2}\right), \tag{52}$$

and the Fourier transform of the elementary function is

$$\hat{f}(u) \propto \sqrt{\hat{G}_0(u,-u)} \propto \exp\left(-\pi^2u^2\frac{a^2}{1+a^2/\sigma_\mathrm{g}^2}\right). \tag{53}$$

Performing the inverse Fourier transform, normalizing the result, and reintroducing $1/a^2$ finally yields the elementary or structural function

$$f(x) = \frac{1}{(\sigma_\mathrm{I}\sigma_\mathrm{g})^{1/2}}\left(\frac{4\sigma_\mathrm{I}^2+\sigma_\mathrm{g}^2}{\pi}\right)^{1/4}\exp\left[-\frac{1}{2}x^2\left(\frac{1}{\sigma_\mathrm{I}^2}+\frac{4}{\sigma_\mathrm{g}^2}\right)\right]. \tag{54}$$

We can see that it is a Gaussian and that its width depends on both σ_I and σ_g. For example, if $\sigma_\mathrm{g}\gg\sigma_\mathrm{I}$, the light is nearly coherent, and the elementary function will be proportional to the field distribution $\sqrt{I_0(x)}$. This implies that only one such function is required to represent the field. On the other hand, if $\sigma_\mathrm{g}\ll\sigma_\mathrm{I}$, the light is nearly incoherent, and the elementary function will become very narrow and resemble a delta function requiring very dense sampling.

It is worth noting that for Gaussian Schell-model beams, $f(x)$ is proportional to $\sqrt{G_0(x,-x)}$. This is not true for all beams.

Continuing the analysis to find the coefficients a_m, we find the intensity

$$I_0(x) = G_0(x,x) = A\exp\left(-\frac{x^2}{\sigma_I^2}\right) \tag{55}$$

and its Fourier transform

$$\hat{I}_0(u) = A\sigma_I\sqrt{\pi}\exp(-\pi^2u^2\sigma_I^2). \tag{56}$$

Furthermore,

$$\widehat{f^2}(u) = \exp\left(\pi^2u^2\frac{\sigma_g^2\sigma_I^2}{4\sigma_I^2+\sigma_g^2}\right) \tag{57}$$

clearly fulfils $\widehat{f^2}(u) \neq 0$, so the quantity

$$\frac{\hat{I}_0(u)}{\widehat{f^2}(u)} = A\sigma_I\sqrt{\pi}\exp\left(-\pi^2u^2\frac{4\sigma_I^4}{4\sigma_I^2+\sigma_g^2}\right) \tag{58}$$

can be formed. This yields the coefficients

$$a_m = \Delta x A\sigma_I\sqrt{\pi}\int_{-1/2\Delta x}^{1/2\Delta x} du \times\exp\left(-\pi^2u^2\frac{4\sigma_I^4}{4\sigma_I^2+\sigma_g^2}\right)\exp(2\pi im\Delta xu). \tag{59}$$

Using the largest possible sampling distance in Eq. (32) yields the sampling distance as

$$\Delta x = \frac{\pi\sigma_I\sigma_g}{\sqrt{(-\ln c)(\sigma_g^2+4\sigma_I^2)}}. \tag{60}$$

Common practice when handling a Gaussian distribution $\exp(-x^2/\sigma^2)$ is often to consider values over a width of 6σ, i.e., for $|x| \leq 3\sigma$, while values for larger $|x|$ are considered to be zero. This would correspond to using $c=\exp(-9) \approx 1.23\times10^{-4}$.

Figure 1 shows the intensities of three Schell-model beams of different degrees of coherence along with the shifted and scaled elementary functions $a_m f^2(x-m\Delta x)$ that add together to form the reconstructed intensity. The reconstructed cross-spectral density is also shown. It is clear that for a high degree of coherence, the elementary functions are wider and a smaller number of modes is needed, while for a low degree of coherence, the elementary functions are narrower and a larger number of modes is needed. In practice, the 6σ criterion might be excessive, and 4σ might be adequate: this and other numerical issues need to be thoroughly studied.

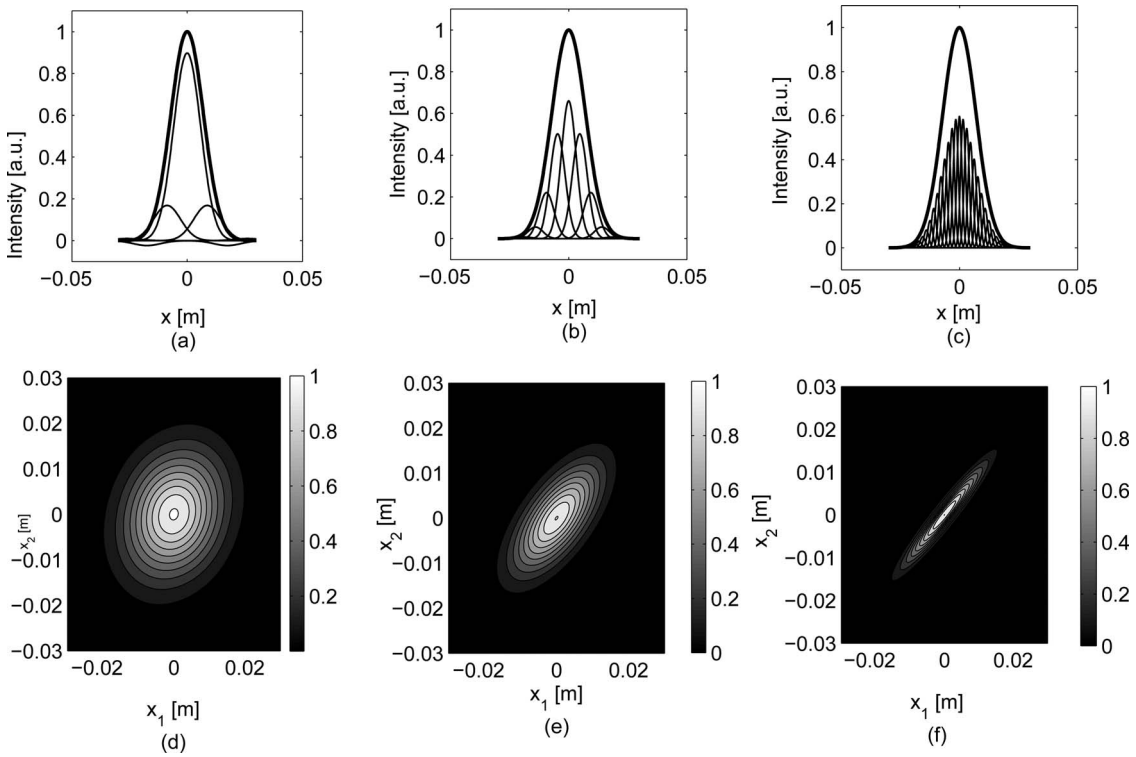

Fig. 1. (a)–(c) Intensity and reconstructed intensity (thick lines; they are almost identical in all three cases) are shown for 1D Gaussian Schell-model beams of different degrees of coherence, along with the scaled and shifted squares of the elementary functions (thin lines). Adding those together gives the reconstructed intensity. The intensity distribution is the same for all beams, with $\sigma_I=0.01$ m, while the coherence varies from high to low as (a) $\sigma_g=0.03$ m with 7 elementary functions required, (b) $\sigma_g=0.01$ m with 13 elementary functions required, and (c) $\sigma_g=0.003$ m with 39 elementary functions required. (d)–(f) Reconstruction of the cross-spectral density for the parameters in (a), (b), and (c) respectively.

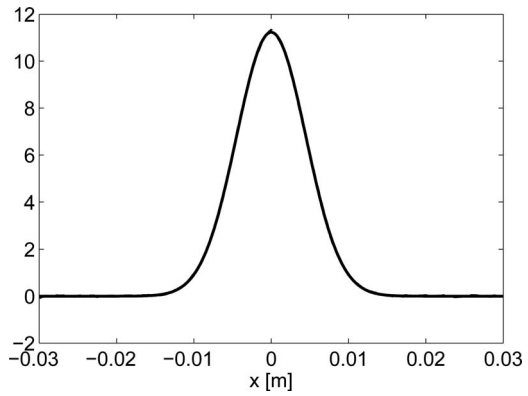

Fig. 2. Elementary function in Eq. (54) (solid curve) and the same function obtained numercially (dashed curve): the two are indistinguishable on the plot.

4. NUMERICAL ANALYSIS

While the theoretical approach applies to most real, 4D cross-spectral densities, the numerical approach requires the assumption of a Schell-model beam,

$$G_0(\mathbf{r}_1,\mathbf{r}_2) = \sqrt{I_0(\mathbf{r}_1)}\sqrt{I_0(\mathbf{r}_2)}\gamma(\mathbf{r}_1-\mathbf{r}_2), \tag{61}$$

where $\gamma(\mathbf{r}_1-\mathbf{r}_2)$ is the degree of coherence. The reason is that we wish to avoid dealing with any 4D quantities, and a Schell-model beam is fully described by the two 2D quantities $I_0(\mathbf{r})$ and $\gamma(\mathbf{r})$. We also assume that the intensity is even, i.e., that $I_0(\mathbf{r})=I_0(-\mathbf{r})$, and for simpler notation we introduce the real and even quantity $U(\mathbf{r}) = \sqrt{I_0(\mathbf{r})}$.

Calculation of the 4D Fourier transform of $\hat{G}_0(\mathbf{u}_1,\mathbf{u}_2)$ is avoided by going directly to the 2D quantity $\hat{G}_0(\mathbf{u},-\mathbf{u})$ in Eq. (22). In Appendix B it is shown that

$$\hat{G}_0(\mathbf{u},-\mathbf{u}) = \int\int_{\infty} \mathrm{d}^2u'\hat{U}^2(\mathbf{u}-\mathbf{u}')\hat{\gamma}(\mathbf{u}') \tag{62}$$

where $\hat{U}(\mathbf{u})$ and $\hat{\gamma}(\mathbf{u})$ are the Fourier transforms of $U(\mathbf{r})$ and $\gamma(\mathbf{r})$, respectively. Note that Eq. (62) requires only 2D functions and 2D integration. As a demonstration, Fig. 2 shows the elementary function displayed earlier in Fig. 1(b), plotted together with the same function obtained numerically. Since this function, obtained only from the initial numerical values of $U(x)$ and $\gamma(x)$, sampled at 500 points, is nearly identical to the theoretically derived version, we can assume that the numerical approach is working well.

5. CONCLUSIONS

We have formalized a method for numerical propagation of partially coherent light, originally presented by Wald *et al.* [15], and clarified its mathematical background. The method produces an elementary function, whose shape is determined by the initial cross-spectral density, and reconstructs the cross-spectral density as an interpolation using shifted and scaled versions of this elementary function. The inherent assumptions and approximations have been outlined and their consequences considered, and a sampling criterion for adequate interpolation has been established. Since the interpolation or sampling functions are not sinc functions or even orthogonal, the traditional Whittaker–Shannon sampling theory cannot be applied, so more recent sampling theory [23] was applied. (Another approach to establish interpolation using nonorthogonal basis functions has been reported by Berry *et al.* [24] and Withington *et al.* [25].) We have also shown that the sampling affects the values of the interpolation or scaling coefficients and thus emphasize the importance of proper sampling.

To illustrate the method, we have found analytically the interpolation for some specific cross-spectral densities. We have also tested the purely numerical approach on these fields and obtained the same results.

The method is of importance in numerical propagation of partially coherent light of relatively high degree of coherence, such as an excimer laser, since it significantly reduces the computer memory required. Further work involves testing of this method for numerical propagation, not just reconstruction of the initial field.

APPENDIX A

The following proof is presented in one dimension but is easily extended to two. To prove that Eq. (24) is fulfilled, we insert the values of the coefficients from Eq. (37) and get

$$\sum_m |a_m|^2 = \sum_m \left(\int_{-1/2\Delta x}^{1/2\Delta x} \mathrm{d}u \frac{\hat{I}_0(u)}{\hat{f}^2(u)} \exp(2\pi i m\Delta x u) \right) \times \left(\int_{-1/2\Delta x}^{1/2\Delta y} \mathrm{d}u' \frac{\hat{I}_0(u')}{\hat{f}^2(u')} \exp(2\pi i m\Delta x u') \right)^*. \tag{A1}$$

Changing the order of integration and summation yields

$$\sum_m |a_m|^2 = \int_{-1/2\Delta x}^{1/2\Delta x} \mathrm{d}u \int_{-1/2\Delta x}^{1/2\Delta y} \mathrm{d}u' \frac{\hat{I}_0(u)}{\hat{f}^2(u)} \frac{\hat{I}_0^*(u')}{\hat{f}^{2*}(u')} \times \sum_m \exp(2\pi i m\Delta x(u-u')), \tag{A2}$$

and recognizing the sum as the Fourier series of a delta function $\delta(\Delta x(u-u'))$ and performing the integration in u' gives

$$\sum_m |a_m|^2 = \frac{1}{\Delta x}\int_{-1/2\Delta x}^{1/2\Delta x} \mathrm{d}u \left| \frac{\hat{I}_0(u)}{\hat{f}^2(u)} \right|^2. \tag{A3}$$

The expression exists if $\hat{I}_0(u)/\hat{f}^2(u)$ is square integrable on $[-1/2\Delta x, 1/2\Delta x]$, which is fulfilled as long as the function has no singularities, i.e., as long as $\hat{f}^2(u)$ has no zeros on the interval. This has already been assumed. Consequently, Eq. (24) is fulfilled for any function of interest.

To prove that the second part of the inequality in Eq. (25) is fulfilled for any physical elementary functions, it is easier to test the equivalent condition in the Fourier domain [23,26]:

$$A \leq \sum_n \sum_m |\hat{\varphi}(\tilde{u}+m,\tilde{v}+n)|^2 \leq B. \tag{A4}$$

By the integral test, this sum exists if $\hat{\varphi}(u,v)$ is square-integrable on $(-\infty,\infty)$, i.e., if $\hat{f}^2(u,v)$ is square-integrable. This will be fulfilled for any physical functions.

APPENDIX B

Taking the Fourier transform of Eq. (61) evaluated at $(\mathbf{u},-\mathbf{u})$ yields

$$\hat{G}_0(\mathbf{u},-\mathbf{u}) = \int\int_\infty \mathrm{d}^2 r_1 \int\int_\infty \mathrm{d}^2 r_2 U(\mathbf{r}_1) U(\mathbf{r}_2) \gamma(\mathbf{r}_1-\mathbf{r}_2) \times \exp(-2\pi \mathrm{i}(\mathbf{r}_1-\mathbf{r}_2)\cdot\mathbf{u}), \tag{B1}$$

where $\mathbf{r}\cdot\mathbf{u}$ represents the scalar product of two vectors. Writing $U(\mathbf{r}_1)$, $U(\mathbf{r}_2)$, and $\gamma(\mathbf{r}_1-\mathbf{r}_2)$ using their Fourier transforms, as, e.g.,

$$U(\mathbf{r}_2) = \int\int_\infty \mathrm{d}^2 u'' \hat{U}(\mathbf{u}'') \exp(2\pi \mathrm{i}\mathbf{u}''\cdot\mathbf{r}_2) \tag{B2}$$

and changing the order of integration yields

$$\begin{aligned}\hat{G}_0(\mathbf{u},-\mathbf{u}) = {} & \int\int_\infty \mathrm{d}^2 u''' \int\int_\infty \mathrm{d}^2 u'' \\ & \times \int\int_\infty \mathrm{d}^2 u' \hat{U}(\mathbf{u}''') \hat{U}(\mathbf{u}'') \hat{\gamma}(\mathbf{u}') \\ & \times \int\int_\infty \mathrm{d}^2 r_1 \exp(2\pi \mathrm{i}(\mathbf{u}'+\mathbf{u}'''-\mathbf{u})\cdot\mathbf{r}_1) \\ & \times \int\int_\infty \mathrm{d}^2 r_2 \exp(2\pi \mathrm{i}(\mathbf{u}''-\mathbf{u}'+\mathbf{u})\cdot\mathbf{r}_2).\end{aligned} \tag{B3}$$

The last two integrations can be performed to yield 2D delta functions $\delta(\mathbf{u}'''+\mathbf{u}'-\mathbf{u})$ and $\delta(\mathbf{u}''-\mathbf{u}'+\mathbf{u})$. Using those to evaluate the integrals in $\mathbf{u}''$ and $\mathbf{u}'''$ yields Eq. (62) if we remember that $\hat{U}(\mathbf{u})$ is real and even [since $U(\mathbf{r})$ is real and even].

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